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CONTENTS

THE SEPARATION OF SUPERSONIC FLOW FROM CURVED PROFILES

1. Introduction and summary of results. The experimental work of A. Ferri [1] has shown that there exists a sharp rise in the pressure on a curved profile (airfoil) in a supersonic stream which is not predicted by the usual theory of pressure distribution. Following this pressure increase, the flow separates from the surface, after which the pressure remains constant or nearly constant over the remainder of the profile. Moreover, the shock which occurs at the rear of the profile is not attached to the end vertex but is separated from the profile and displaced in the direction of the stream. It is felt that the existence of this shock is directly connected with the phenomenon of separation. If this view is correct, the problem of the separation of supersonic flow from curved profiles is definitely a problem *in the large*. The above facts are indicated in Fig. 1 in which AB represents the small interval of increasing pressure. Separation occurs at point B, the shock at the rear of the profile is represented by CD, and CE is a stream line behind the shock. Between point B and the tail vertex T the pressure is practically constant.

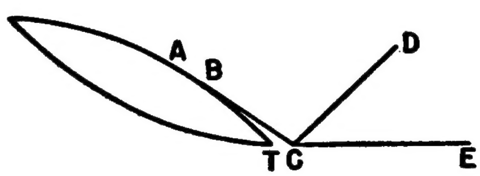
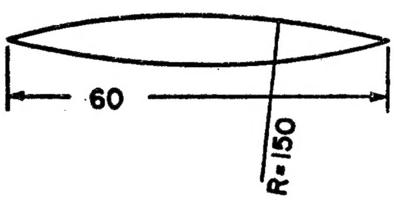


Fig. 1

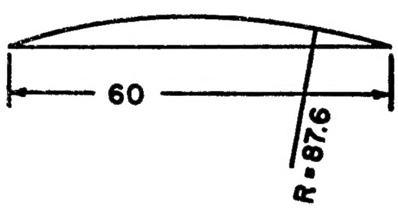
As a consequence of the above behavior of the pressure function, both immediately before and following the point of separation of the flow, the experimental lift and drag on the profile differ markedly from the theoretical values obtained by the standard pressure calculations. A theory which will account for the actual pressure effects with good quantitative accuracy would appear to be not merely of academic interest but of considerable practical importance in the design of supersonic airfoils. In spite of this incentive, there has been no attempt in the literature to

solve this problem, although the need for its solution was recognized in this country in 1945 in a report prepared by Edmonson, Kennedy and Nering for the Applied Physics Laboratory at The Johns Hopkins University.

Undoubtedly the problem of the separation of supersonic flow from an airfoil and the determination of the associated pressure effects would have been solved before now except for the difficulties inherent in its mathematical formulation. What is apparently needed is a guiding principle which will lead to the establishment of a suitable mathematical model. We have attempted to meet this situation by assumptions on the *tendency* of mach angles to remain invariant during the separation process (§13). One consequence of these assumptions is that a certain element of *rigidity* is imposed on the flow pattern. The essential physical validity of the assumptions is indicated by the excellent agreement between the experimental and calculated pressure graphs for the $GU2$ and $GU3$ profiles over a wide range of angles of attack (§19). The cross section of the $GU2$ profile consists of two circular arcs while that of the



$GU2$ PROFILE
FIG. 2



$GU3$ PROFILE
FIG. 3

$GU3$ profile is formed by a circular arc and its secant. These profiles are shown in Figs. 2 and 3 with their relative dimensions. Calculations are carried out for the $GU2$ profile for $M = 2.13$ and for the $GU3$ profile for $M = 1.85$ and $M = 2.13$ where M denotes the mach number of the free stream. These calculations specifically yield the *final pressure* on the profile (§15), the *point of separation* (§16) and the *back pressure interval* over which the pressure increases abruptly (§17).

It is possible that deviations from the assumed rigidity of the flow pattern following separation occur when the mach number M is decreased sufficiently in the case of any given profile. Evidence to this effect is to be found in the experimental pressure measurements on the $GU2$ profile for $M = 1.85$, which show a significant increase beyond the separation point, whereas in the case of the same profile for $M = 2.13$

(as well as the profile $GU3$ for $M = 1.85$ and 2.13) constant values of the pressure are attained. On the other hand, there is the possibility that this observed pressure increase following separation of flow was the result of conditions within the wind tunnel which were measurably effective only for the lower mach number $M = 1.85$ in the case of the $GU2$ profile. In this connection it may be mentioned that Ferri has stated in his article [1] that the apparatus used was far from perfect and that its final form had not been decided upon at the time of the experiments although these experiments were never repeated with improved apparatus. In view of the questions here raised, separation calculations for the $GU2$ profile with $M = 1.85$ have not been included.

The standard method of pressure calculation lies at the basis of this theory of the separation of supersonic flow. This has made it almost necessary to give some account of the standard theory. Since the usual derivations are in part geometrical in character, we have taken this opportunity to present a completely analytical discussion. It is hoped that this modification of the usual treatment will be of some interest in itself.

A comparison of the calculated and experimental pressure graphs shows that the differences between the final calculated and experimental pressures are, in general, almost precisely equal to the differences in the values of these quantities at the point at which the back pressure begins effectively to develop. This suggests strongly that an improvement in the standard procedure for pressure calculation would automatically lead to a corresponding improvement in the determination of the value of the pressure after separation. In order to test this hypothesis we have extended the experimental pressure graph for the $GU3$ profile for $M = 2.13$ and angle of attack $a = 0$ in a natural way as shown in the figure at the top of the graph in §20. The curve so obtained was then used in place of the complete pressure graph, calculated by the standard procedure, for the determination of separation effects. As a result of this calculation we arrive at a pressure graph which passes accurately through all experimental pressure values (§20), thus verifying the above hypothesis.[1]

2. Stationary ideal gases. Consider the equations governing the flow of a stationary ideal gas with viscosity and thermal conductivity zero, namely

$$p_{,\alpha} + \rho u_\sigma u_{\alpha,\sigma} = 0 \text{ (equations of motion)}, \tag{1}$$

$$\rho_{,\alpha} u_\alpha + \rho u_{\sigma,\sigma} = 0 \text{ (equation of continuity)}, \tag{2}$$

[1] The drawings and graphs were made by D. M. Nead in his capacity of research assistant under Navy Contract N6onr-180, Task Order V, with Indiana University, and the computations were carried out by B. H. McCandless, a graduate student in the Department of Mathematics.

in which p, ρ and u_α denote, as is customary, the pressure, density and velocity components. We assume the motion referred to a system of rectangular coordinates x^α; then the comma in the above and following equations represents partial differentiation. In addition there is the equation which states that the entropy S is constant along stream lines. This condition when combined with the equations (1) results in the relation

$$\frac{\gamma p}{(\gamma - 1)\rho} + \frac{1}{2} u_\sigma u_\sigma = H \qquad \text{(energy equation)}, \tag{3}$$

where γ is the ratio of the two specific heats c_p and c_v (assumed constant) and H is constant along stream lines but can vary from stream line to stream line (weak Bernoulli equation). Conversely the condition that S be constant along stream lines can be recovered from (1) and (3). Hence the general equations determining the motion of the gas under consideration are given completely by (1), (2) and (3).

In the following discussion we shall assume that H is an absolute constant, i.e., its value is independent of the stream line (strong Bernoulli condition). It can be supposed, moreover, that the values of the constants γ and H are known from the conditions of the problem. Then the number of equations in the set (1), (2) and (3) is exactly equal to the number of functions u_α, p and ρ to be determined.

Remark 1. The quantity in the left member of (3) is invariant in the transition across a shock wave, i.e., its value is the same at adjacent points on the two sides of a shock surface. It follows that the value of the constant H is not effected by shocks which occur when an obstacle is placed in a uniform supersonic gas flow, i.e., it has the same constant value over the entire field of flow. The above assumption, namely that H is an absolute constant, can therefore be made in most practical applications involving shocks.

By means of (3), the density ρ and its derivatives $\rho_{,\alpha}$ can be eliminated from (1) and (2). The resulting equations can be written in the form

$$c^2 p_{,\alpha} + \gamma p u_\sigma u_{\alpha,\sigma} = 0, \tag{4}$$

$$p_{,\alpha} u_\alpha + \gamma p u_{\sigma,\sigma} = 0, \tag{5}$$

where $c^2 = \gamma p/\rho$ is the velocity of sound in the gas. From (3) the quantity c^2 is given explicitly in terms of the velocity components u_α by

$$c^2 = \frac{\gamma - 1}{2}(2H - u_\sigma u_\sigma). \tag{6}$$

Conversely, from (4), (5) and (6), the equations (1), (2) and (3) can be recovered. Hence in our discussion of the flow problem we can limit our

attention to the equations (4) and (5) in the variables p and u_α with c^2 given by (6). When these equations have been solved for the functions p and u_α, the density ρ will be determined from (6), and the expression $\gamma p/\rho$ for c^2.

Remark 2. Multiply (5) by c^2, equation (4) by u_α, and then subtract corresponding members of these equations to obtain

$$u_{\alpha,\beta}u_\alpha u_\beta = c^2 u_{\sigma,\sigma}. \tag{7}$$

This relation expresses the condition that the entropy is constant along stream lines (see, T. Y. Thomas, *The fundamental hydrodynamical equations and shock conditions for gases*, Math. Mag., vol. 22, pp. 169–189).

3. Characteristic curves for supersonic flow. In the following we shall limit our attention to plane flow. Let C be a curve in the coordinate plane defined parametrically by $x^\alpha(s)$. We suppose that the functions $x^\alpha(s)$ are continuous and differentiable and that $\dot{x}^\alpha\dot{x}^\alpha>0$ where the "dot" is used to denote differentiation with respect to the parameter s. Now define functions $u_\alpha(s)$ and $p(s)$ along C, and let us raise the following question: *Will there exist a compressible flow of the generality considered in §2 having the velocity components $u_\alpha(s)$ and pressure $p(s)$ along C?* We shall be concerned primarily with the discussion of this question under the condition that the flow is supersonic.

As necessary conditions for the existence of the above flow, equations (4) and (5) must be satisfied along C. There are also conditions on the derivatives $u_{\alpha,\beta}$ and $p_{,\alpha}$ which result from differentiation of the given functions $u_\alpha(s)$ and $p(s)$ with respect to the parameter s. Combining these relations we have the following system

$$\begin{cases} c^2(\log p)_{,\alpha} + \gamma u_\sigma u_{\alpha,\sigma} = 0 \\ (\log p)_{,\sigma}u_\sigma + \gamma u_{\sigma,\sigma} = 0 \\ u_{\alpha,\sigma}\dot{x}^\sigma = \dot{u}_\alpha \\ (\log p)_{,\sigma}\dot{x}^\sigma = \dot{p}/p \end{cases} \tag{8}$$

where c^2 is given by (6). These equations involve as unknowns the functions u_α and p alone. As explained in §2, any solution of the first two equations (8) will determine a compressible flow with density ρ given by $c^2 = \gamma p/\rho$ and the relation (6).

The system (8) is represented completely by Table 1 in which the entries are the coefficients of the derivatives $u_{\alpha,\beta}$ and $(\log p)_{,\alpha}$ as indicated in the first row of the table. The last column labelled *R.M.* contains the right members of the system (8). Each entry in this table is determined by the curve C and the data $u_\alpha(s)$ and $p(s)$ assigned along C.

Denote by D the sixth order determinant whose elements are obtained from the first six columns of this table. If $D \neq 0$ on C, then the deriva-

TABLE 1

$u_{1,1}$	$u_{1,2}$	$u_{2,1}$	$u_{2,2}$	$(\log p)_{,1}$	$(\log p)_{,2}$	*R.M.*
γu_1	γu_2	0	0	c^2	0	0
0	0	γu_1	γu_2	0	c^2	0
γ	0	0	γ	u_1	u_2	0
$\dot{x}^1$	$\dot{x}^2$	0	0	0	0	u_1
0	0	$\dot{x}^1$	$\dot{x}^2$	0	0	u_2
0	0	0	0	$\dot{x}^1$	$\dot{x}^2$	$\dot{p}/p$

tives $u_{\alpha,\beta}$ and $p_{,\alpha}$ have a unique determination along C. Under this condition it is a theorem in differential equations that there exists one, and only one, compressible flow, i.e., solution $u_\alpha(x)$ and $p(x)$ of the equations (4) and (5), which assumes the assigned values $u_\alpha(s)$ and $p(s)$ on C.

Suppose now that $D = 0$ at each point of C. A curve for which this is the case is called a *characteristic curve of the system* (4) *and* (5), *more fully a characteristic curve relative to the data $u_\alpha(s)$ and $p(s)$ assigned along the curve.* Equating D to zero and simplifying the equation, we find that the conditions for a characteristic curve are

$$(9)\quad (u_2\dot{x}^1 - u_1\dot{x}^2)[(u_1^2 - c^2)(\dot{x}^2)^2 - 2u_1u_2\dot{x}^1\dot{x}^2 + (u_2^2 - c^2)(\dot{x}^1)^2] = 0.$$

The conditions (9) can be written in more compact form as follows. Denote by λ^α the components of the unit tangent vector to C, and by ν^α the components of the unit normal to this curve; then

$$\dot{x}^1 \sim \lambda^1 = -\nu^2, \dot{x}^2 \sim \lambda^2 = \nu^1.$$

Making these substitutions in (9) we obtain

$$(10)\qquad u_\sigma\nu^\sigma(u_\alpha u_\beta - \delta_{\alpha\beta}c^2)\nu^\alpha\nu^\beta = 0$$

as the conditions for the characteristic curve. Another form of these conditions which may be considered involves the slope of the curve C. If we denote the slope by θ, so that $\dot{x}^2 = \theta\dot{x}^1$, the characteristic conditions (9) yield

$$(11a)\qquad u_2 - \theta u_1 = 0, \quad \text{or}$$

$$(11b)\qquad (u_1^2 - c^2)\theta^2 - 2u_1u_2\theta + (u_2^2 - c^2) = 0.$$

Since the curves having slope u_2/u_1 are stream lines, we see immediately from (11a) that *the stream lines form a family of characteristic curves.*

The condition that (11b) have a real solution θ on C is that $c^2(v^2 - c^2) \geqq 0$ where $v^2 = u_\sigma u_\sigma$ is the square of the velocity. Hence the data $u_\alpha(s)$ and $p(s)$, assigned on C, must be such that this inequality is satisfied if

C is to be a characteristic in virtue of (11b). Now suppose that the data u_α and p on C is taken to be that resulting from a given supersonic flow so that the condition $c^2(v^2 - c^2) > 0$ is automatically satisfied. When the data on a characteristic curve are obtained in this way, the curve will be said to be *determined* by the given flow. Hence, *in addition to the stream lines, there will be two real families of characteristic curves determined by a given supersonic flow having slopes θ which satisfy* (11b). If we denote by $\theta = \xi$ and $\theta = \zeta$ the two solutions of (11b) for supersonic flow and solve this equation by the quadratic formula, we obtain

(12) $$\theta = \xi = \frac{u_1 u_2 + c\sqrt{v^2 - c^2}}{u_1^2 - c^2},$$

(13) $$\theta = \zeta = \frac{u_1 u_2 - c\sqrt{v^2 - c^2}}{u_1^2 - c^2},$$

where we have identified the slopes ξ and ζ by means of these relations.[2] Since $v^2 > c^2$ for supersonic flow, the curves having slopes ξ and ζ are distinct as implied in the above italicized statement.

4. A property of the characteristic curves. Let us denote, for brevity, by u_2/u_1, ξ and ζ the families of the characteristics determined by a supersonic flow which have slopes u_2/u_1, ξ and ζ, respectively. It will be

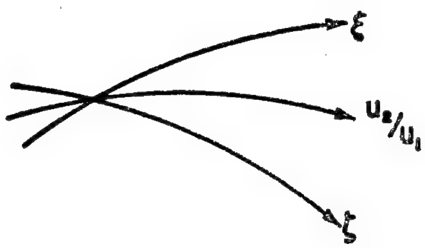


FIG. 4

assumed that the curves ξ and ζ are so directed that they are divided *internally* by the stream line u_2/u_1 as shown in Fig. 4. We now prove the following result: *The characteristic line of the family u_2/u_1 (stream line) through any point P bisects the angle formed by the characteristic lines of the families ξ and ζ through P.* To prove this it suffices to show that

[2] The occurrence of the quantity $u_1^2 - c^2$ in the denominator in the above equations (12) and (13) implies that the coordinate axes are so oriented that this quantity does not vanish in the region under consideration.

$$\frac{\xi - u_2/u_1}{1 + u_2\xi/u_1} = \frac{u_2/u_1 - \zeta}{1 + \zeta u_2/u_1}, \tag{14}$$

since the left member of this equation is the tangent of the angle formed by the lines of the families u_2/u_1 and ξ, while the right member is the tangent of the angle formed by the lines of the families ζ and u_2/u_1.

It can be assumed that the axes are so oriented that $u_1 \neq 0$ and and $u_2 \neq 0$ at P. Also neither denominator in (14) can vanish, for the vanishing of either denominator would mean that a solution of (11b) would be given by $\theta = -u_1/u_2$; but making this substitution into (11b) the resulting equation reduces to $v^2(v^2 - c^2) = 0$, which is not satisfied for supersonic flow. Equation (14) can now be written in the form

$$(u_1^2 - u_2^2)(\xi + \zeta) + 2u_1u_2\xi\zeta - 2u_1u_2 = 0. \tag{15}$$

But if we substitute into (15) the values of ξ and ζ given by (12) and (13), the equation will be satisfied identically. Hence (14) holds and the above assertion is proved.

Denote by $\mu > 0$ the positive angle, not exceeding 90°, which the stream line makes with either characteristic ξ or ζ. Also denote by ω the inclination of the stream lines and by β the inclination of the ξ characteristic so that $\mu = \beta - \omega$ (see Fig. 4). Then

$$u_1 = v \cos \omega, \; u_2 = v \sin \omega$$

$$\dot{x}^1 \sim \cos \beta, \; \dot{x}^2 \sim \sin \beta.$$

Hence equation (9) yields

$$c^2 = v^2(\sin \beta \cos \omega - \cos \beta \sin \omega)^2 = v^2 \sin^2 \mu. \tag{16}$$

The equation $m^2 = v^2/c^2$ defines the *local mach number* m. Introducing this number into the relation (16), we can write $\sin \mu = 1/m$. *Hence at at any point of the flow the relation* $\sin \mu = 1/m$ *holds between the local mach number* m *and the positive inclination* μ *of the* ξ *or* ζ *characteristic relative to the stream line.*

5. Consistency relations for characteristic curves. Consider Table 1 in §3 which was used in the definition of the characteristic curves C. Suppose the coordinate axes oriented so that at a given point P of the characteristic curve $x^\alpha(s)$ we have $\dot{x}^1 \neq 0$ and $\dot{x}^2 \neq 0$. It follows that $\theta \neq 0$ at P. Then when we substitute $\dot{x}^2 = \theta\dot{x}^1$ in Table 1 and take due regard to the last column labelled *R.M.*, it is seen that the quantity $\dot{x}^1$

can be cancelled without changing the rank of any matrix appearing in the table.[3] The matrix determined by the table is now given by

$$
(17) \qquad \begin{Vmatrix} \gamma u_1 & \gamma u_2 & 0 & 0 & c^2 & 0 & 0 \\ 0 & 0 & \gamma u_1 & \gamma u_2 & 0 & c^2 & 0 \\ \gamma & 0 & 0 & \gamma & u_1 & u_2 & 0 \\ 1 & \theta & 0 & 0 & 0 & 0 & \dot{u}_1 \\ 0 & 0 & 1 & \theta & 0 & 0 & \dot{u}_2 \\ 0 & 0 & 0 & 0 & 1 & \theta & \dot{p}/p \end{Vmatrix}.
$$

Equations (11) in §3 which give the conditions for the characteristic curves in terms of the slope θ result by equating to zero the determinant of order six which is obtained from the elements of the first six columns of this matrix.

Now consider the following determinant δ selected from the above matrix, namely

$$
\delta = \begin{vmatrix} \gamma u_2 & 0 & 0 & c^2 & 0 \\ 0 & \gamma u_1 & \gamma u_2 & 0 & c^2 \\ 0 & 0 & \gamma & u_1 & u_2 \\ \theta & 0 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 & 0 \end{vmatrix}.
$$

Expanding δ we obtain

$$
(18) \qquad \delta = -\gamma c^2 \theta [(u_2^2 - c^2) - u_1 u_2 \theta].
$$

Hence $\delta = \gamma c^4 \theta$ when $\theta = u_2/u_1$ and consequently $\delta \neq 0$ in the neighborhood of the point P on a characteristic stream line.

Suppose now that θ satisfies (11b). Then if the bracket expression in (18) vanishes we have

$$
\theta = (u_2^2 - c^2)/u_1 u_2 .
$$

Substituting this value of θ into (11b), we find that the resulting equation reduces to $u_1^2 + u_2^2 = c^2$, which is not satisfied for supersonic flow. Hence again $\delta \neq 0$. It follows in all cases that δ is different from zero, and hence the matrix (17) will have rank not less than the order of the determinant δ, i.e., the rank of the matrix (17) cannot be less than 5.

[3] The rank of the complete matrix determined by Table 1 is obviously independent of rotations of coordinate axes on account of its relationship to the equations (8), but the rank of certain matrices selected from the table may depend on the choice of coordinate axes.

Next consider the sixth order determinant

$$\Delta = \begin{vmatrix} \gamma u_2 & 0 & 0 & c^2 & 0 & 0 \\ 0 & \gamma u_1 & \gamma u_2 & 0 & c^2 & 0 \\ 0 & 0 & \gamma & u_1 & u_2 & 0 \\ \theta & 0 & 0 & 0 & 0 & \dot{u}_1 \\ 0 & 1 & \theta & 0 & 0 & \dot{u}_2 \\ 0 & 0 & 0 & 1 & \theta & \dot{p}/p \end{vmatrix}$$

which appears on the right of the matrix (17). Along a characteristic curve $\Delta = 0$ since otherwise the equations (8) would not be consistent. But $\Delta = 0$ and $\delta \neq 0$ insures that the matrix (17) has rank 5 at points of the characteristic, and this suffices for the algebraic consistency of the equations (8) as equations for the determination of the derivatives $u_{\alpha,\beta}$ and $p_{,\alpha}$ along the characteristic curve.

Expanding Δ we find

$$\begin{aligned} -c^2\theta[(u_2^2 - c^2) - u_1u_2\theta]\frac{\dot{p}}{p} \qquad\qquad & \\ + \gamma u_2[c^2 - (u_2 - \theta u_1)^2]\dot{u}_1 + \gamma c^2 u_1 \dot{u}_2 \theta^2 &= 0. \end{aligned} \tag{19}$$

If θ satisfies (11a) equation (19) reduces to

$$c^2 \frac{\dot{p}}{p} + \gamma u_\sigma \dot{u}_\sigma = 0. \tag{20}$$

Now suppose that θ satisfies (11b). Then from this equation we have

$$-[(u_2^2 - c^2) - u_1u_2\theta] = [(u_1^2 - c^2)\theta - u_1u_2]\theta,$$

and

$$\begin{aligned} c^2 - (u_2 - \theta u_1)^2 &= -[(u_2^2 - c^2) - 2u_1u_2\theta + u_1^2\theta^2] \\ &= -[(u_1^2 - c^2)\theta^2 - 2u_1u_2\theta + (u_2^2 - c^2) + c^2\theta^2] = -c^2\theta^2. \end{aligned}$$

Making these substitutions in (19) we are led to the following equation

$$[(u_1^2 - c^2)\theta - u_1u_2]\frac{\dot{p}}{p} + \gamma(u_1\dot{u}_2 - u_2\dot{u}_1) = 0. \tag{21}$$

Equation (20) is invariant under rotations of the coordinate axes and hence is independent of the orientation of axes which was assumed at the beginning of this discussion. Equation (21) is likewise seen to be invariant when account is taken of the condition determining the slope θ of the characteristic curves, i.e., the equation (11b). Thus from (12) and (13) we can write

$$(u_1^2 - c^2)\theta - u_1u_2 = c\sqrt{v^2 - c^2} \qquad (\theta = \xi),$$
$$(u_1^2 - c^2)\theta - u_1u_2 = -c\sqrt{v^2 - c^2} \qquad (\theta = \zeta).$$

Hence, according as $\theta = \xi$ or $\theta = \zeta$ equation (21) becomes[4]

$$c\sqrt{v^2 - c^2}\,\frac{\dot{p}}{p} + \gamma u_\alpha \dot{u}_\beta e_{\alpha\beta} = 0 \qquad (\theta = \xi), \tag{22}$$

$$c\sqrt{v^2 - c^2}\,\frac{\dot{p}}{p} - \gamma u_\alpha \dot{u}_\beta e_{\alpha\beta} = 0 \qquad (\theta = \zeta). \tag{23}$$

The invariance of (22) and (23) under rotations of coordinates is evident. We can now state the following result which is independent of the coordinate system employed. *Equation (20) must be satisfied along the characteristics given by $\theta = u_2/u_1$ (stream lines), and (22) and (23) along the characteristics ξ and ζ respectively.* When (20) is satisfied at points of a characteristic stream line, or when (22) or (23) is satisfied at points of a ξ or ζ characteristic, then the system (8) will be algebraically consistent as equations for the determination of the derivatives $u_{\alpha,\beta}$ and $p_{,\alpha}$ along the characteristic.

Remark. We now give another form of the equations (22) and (23) which will be useful in the following discussion. By differentiation of the relations $u_1 = v \cos \omega$ and $u_2 = v \sin \omega$ and substitution into the last term of (22) or (23), we find that

$$u_\alpha \dot{u}_\beta e_{\alpha\beta} = u_1\dot{u}_2 - u_2\dot{u}_1 = v^2\dot{\omega}.$$

Making this substitution into (22) and (23) and introducing the local mach number m, these conditions become

$$\sqrt{m^2 - 1}\,\frac{\dot{p}}{p} + \gamma m^2\dot{\omega} = 0 \qquad (\theta = \xi), \tag{22a}$$

$$\sqrt{m^2 - 1}\,\frac{\dot{p}}{p} - \gamma m^2\dot{\omega} = 0 \qquad (\theta = \zeta). \tag{23a}$$

The following result is an immediate consequence of (22a) and (23a): *If the inclination ω of the stream lines is constant along a ξ or ζ characteristic, so also is the pressure p.*

6. Rotational and irrotational flow. In this section we consider certain special relations which are consequences of the equations of the preceding sections. We first recall the definitions of a few terms which are basic in the discussion.

[4] The quantities $e_{\alpha\beta}$ defined by $e_{12} = -e_{21} = 1$ and $e_{11} = e_{22} = 0$ are the components of a skew symmetric tensor under proper orthogonal rotations of the coordinate axes.

A flow is said to be *rotational* if the rotation, i.e., the vector having components $e^{\alpha\beta\gamma}u_{\beta,\gamma}$ in rectangular coordinates, does not vanish identically.[5] When this vector vanishes the flow is called *irrotational.* The flow is said to be *non-isentropic,* or sometimes *non-adiabatic,* if the entropy S is not constant throughout the flow. Otherwise the flow is said to be *isentropic* or *adiabatic.* The condition of isentropy can be expressed by the statement that the quantity N, which occurs in the relation $p = N\rho^\gamma$, is an absolute constant; for non-isentropic flow N is constant only along stream lines and varies from stream line to stream line.[6]

In the case of plane flow, the rotation is determined by the single component

$$\phi = u_{1,2} - u_{2,1} .$$

We shall now show that *plane irrotational flow is isentropic and conversely.* It follows that *plane rotational flow is non-isentropic and conversely.* In other words, irrotational and isentropic flow, and likewise rotational and non-isentropic flow, are equivalent in the plane case under consideration.

First differentiate (3) with respect to x^α to obtain

$$\frac{\gamma}{\gamma - 1}\left[\frac{p_{,\alpha}}{\rho} - \frac{p}{\rho^2}\rho_{,\alpha}\right] + u_\sigma u_{\sigma,\alpha} = 0. \tag{24}$$

Assuming the flow irrotational (1) becomes

$$\frac{p_{,\alpha}}{\rho} + u_\sigma u_{\sigma,\alpha} = 0. \tag{25}$$

Subtracting corresponding members of these two equations we have

$$\left(\frac{\gamma}{\gamma - 1} - 1\right)\frac{p_{,\alpha}}{\rho} - \frac{\gamma}{\gamma - 1}\frac{p}{\rho^2}\rho_{,\alpha} = 0. \tag{26}$$

But this last equation can be integrated readily to give $N_{,\alpha} = 0$ where N is the ratio p/ρ^γ. Hence N is an absolute constant and the flow is isentropic in accordance with the above definition. Now assume $p = N\rho^\gamma$, with N an absolute constant. But under this condition we can deduce (26). Subtracting corresponding members of (24) and (26), we obtain (25). But when (25) is combined with (1) we obtain

$$u_\sigma(u_{\alpha,\sigma} - u_{\sigma,\alpha}) = 0.$$

Hence $u_2\phi = 0$ and $u_1\phi = 0$. It follows that $\phi = 0$ at any point where the velocity does not vanish. From continuity $\phi = 0$ at any point, which is

[5] The quantities $e^{\alpha\beta\gamma}$ with $e^{123} = 1$ are the components of a completely skew symmetric tensor in three dimensional space under proper rotations of the rectangular coordinate axes.

[6] The relation $p = N\rho^\gamma$, in which N is constant along stream lines, is equivalent to the condition, stated in §2, that the entropy S is constant along stream lines.

a limit of points for which the velocity does not vanish. Also $\phi = 0$ identically in any open region in which the velocity is equal to zero. Hence $\phi = 0$ at all points of the flow, which means that the flow is irrotational. This proves the above italicized statements.

In the case of plane flow it can be shown that [2]

$$\phi = Ap, \tag{27}$$

where the quantity A is defined by

$$(\gamma - 1)NA = -dN/d\psi \tag{28}$$

with ψ the stream function. Hence A is constant along stream lines but varies from stream line to stream line for rotational flow. The relation (27) is first proved subject to the condition that the velocity does not vanish, but this condition is easily removed by the type of argument employed above. Thus, if $v = 0$ at a point P, and P is a limit of points at which $v \neq 0$, we see that (27) is satisfied at P by continuity. In case $v = 0$ at P, and P lies in an open set of points at which $v = 0$, then ϕ vanishes identically in this open set; but $\phi \equiv 0$ means that the flow is irrotational, and hence the flow is isentropic in this open set by the above result. It follows from the equation (28) defining A that A must vanish in this open set. Hence both members of (27) vanish in the open set and this equation is again satisfied at the point P. *The equation* (27) *is therefore satisfied throughout any region in which the flow is continuous and differentiable.*

Remark 1. *Plane flow is irrotational* (*and hence isentropic*) *if, and only if*, $A = 0$. This is an immediate consequence of the above results.

Remark 2. If the inclination ω of the stream lines is constant along a ξ or ζ characteristic in plane flow, then the pressure p was shown to be constant along this characteristic (see *Remark* at the end of §5). When, in particular, the flow is irrotational, so that N is an absolute constant, we see from the equation $p = N\rho^\gamma$ that ρ is also constant along the characteristic. It follows from these results and the energy equation (3) that the velocity v and hence the velocity components u_α are likewise constant. In addition, the velocity of sound c, given by $c^2 = \gamma p/\rho$, and the local mach number m, where $m = v/c$, must be constant. Similarly the quantity μ which gives the inclination of the characteristic relative to the stream lines must be constant in view of the relation $\sin \mu = 1/m$. But from the fact that μ and ω are constant, it follows that the inclination of the characteristic is constant, and hence the characteristic is a straight line. We combine these results briefly in the following italicized statement: *If, in the case of plane irrotational flow, the inclination ω of the stream lines is constant along a ξ or ζ characteristic, then the characteristic is a straight line having constant inclination μ relative to the stream lines and along which the pressure p, density ρ, velocity vector and its components u_α, velocity of sound c and local mach number m are constant.*

7. The fundamental differential relation. Let σ denote arc length along a stream line measured in the direction of the flow. Also let us denote by n distance in the direction of a vector η normal to the velocity vector v; we suppose the normal η so directed that the vector pair v, η has the same orientation as the coordinate axes. The following formulas [2]

$$\frac{dv}{dn} = vK + \phi, \tag{29}$$

$$\frac{d\omega}{dn} = \frac{m^2 - 1}{v} \frac{dv}{d\sigma}, \tag{30}$$

can now be derived, where $K = d\omega/d\sigma$ is the curvature of the stream line. The first of these relations is a kinematical identity while the second is a consequence of the equations of §2 which determine the flow. In addition let us note the relation

$$\tan \mu = \frac{1}{\sqrt{m^2 - 1}} \tag{31}$$

which is equivalent to the equation $\sin \mu = 1/m$ derived in §4.

Now let s be arc length along a ξ or ζ characteristic as shown in Fig. 5.

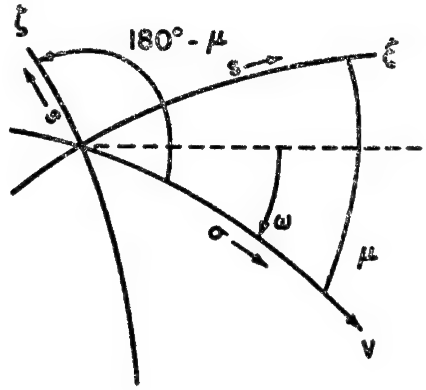


Fig. 5

With reference to the ξ characteristic we then have

$$\frac{dv}{ds} = \cos \mu \frac{dv}{d\sigma} + \sin \mu \frac{dv}{dn},$$

$$\frac{d\omega}{ds} = \cos \mu \frac{d\omega}{d\sigma} + \sin \mu \frac{d\omega}{dn}.$$

Making the substitutions (29) and (30) into these equations and dividing corresponding members, we arrive at the following relation

$$\frac{1}{v}\frac{dv}{d\omega} = \frac{dv/d\sigma + (vK + \phi) \tan \mu}{vK + (m^2 - 1) \tan \mu \, dv/d\sigma}$$

for the derivative of the velocity v with respect to the inclination ω of the stream lines along a ξ characteristic. When account is taken of the relation (31) the above equation becomes

$$\frac{1}{v}\frac{dv}{d\omega} = \frac{[dv/d\sigma + (vK + \phi) \tan \mu] \tan \mu}{dv/d\sigma + vK \tan \mu}. \tag{32}$$

Equation (32) is seen to simplify considerably in the case of irrotational flow for which $\phi = 0$. Assuming that the flow is irrotational we have

$$\frac{1}{v}\frac{dv}{d\omega} = \tan \mu \qquad \text{(along the } \xi \text{ characteristic).} \tag{33}$$

It is clear from Fig. 5 that the corresponding relation for the ζ characteristic is obtainable from (33) by replacing μ by $180° - \mu$. Making this substitution in (33) we thus have

$$\frac{1}{v}\frac{dv}{d\omega} = -\tan \mu \qquad \text{(along the } \zeta \text{ characteristic).} \tag{34}$$

The equations (33) and (34) are fundamental in the derivation of the method which we shall employ for calculating the pressure, density and velocity along the profile.

8. Relation between the inclination of the profile and the characteristic directions. It will be assumed that the contour of the profile under consideration is *regular* in the sense that it is defined by functions $x^\alpha(t)$ having at each point of the contour the following properties. *First*, the functions $x^\alpha(t)$ are continuous and possess continuous first derivatives and *second*, the condition $\dot{x}^\alpha\dot{x}^\alpha > 0$ holds, where the "dot" denotes differentiation with respect to t. Under these conditions the arc length σ of the contour can be defined, the contour will have a continuously turning tangent, and the curvature $K = d\omega/d\sigma$, where ω is the inclination of the contour, will exist and be a continuous function of the arc length. It will furthermore be assumed that along the upper contour of the profile the inclination ω is a *decreasing* function of the arc length σ, where $\sigma = 0$ at the front vertex V and $\sigma = L$ at the tail vertex T so that L is the length of the upper contour. In the following we shall deal specifically with the upper contour of the profile. By reflecting the lower contour

about a straight line in the direction of the undisturbed flow, a curve will be obtained for which it will be assumed that the above conditions are valid; hence the discussion can be applied to the reflected contour to yield results which can be transferred to the lower contour.

Let us now substitute the quantity $c^2 = \gamma p/\rho$ into the energy equation (3) to obtain

$$\frac{c^2}{\gamma - 1} + \frac{v^2}{2} = H. \tag{35}$$

From this equation we can eliminate c^2 by means of the relation $c = v \sin \mu$ derived in §4; solving the resulting equation for v^2 and also for $\sin^2\mu$ we have

$$v^2 = \frac{2(\gamma - 1)H}{\gamma - 1 + 2\sin^2 \mu}, \tag{36}$$

$$\sin^2 \mu = \frac{(\gamma - 1)H}{v^2} - \frac{\gamma - 1}{2}. \tag{37}$$

We now consider (36) and (37) along a ξ or ζ characteristic for which the inclination ω of the stream lines can be introduced as a parameter. Then differentiating (37) with respect to ω, we are led to the following equation

$$\sin \mu \cos \mu \frac{d\mu}{d\omega} = \frac{-(\gamma - 1)H}{v^2} \cdot \frac{1}{v}\frac{dv}{d\omega}. \tag{38}$$

We can now eliminate the second factor in the right member of (38) by means of (33) or (34), depending on whether we are dealing with a ξ or a ζ characteristic. Making this elimination and using (36), we arrive at a differential equation which can be written in the form

$$d\omega = \frac{\pm\, d\mu}{1 - k^2 \sec^2 \mu}, \tag{39}$$

where $k^2 = (\gamma + 1)/2$; here the plus sign applies in the case of a ξ characteristic and the minus sign for a ζ characteristic. It can easily be verified that a solution of (39) is given by

$$\pm\, \omega = \text{const.} - f(\mu), \tag{40}$$

where

$$f(\mu) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1}\left(\sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan \mu\right) - \mu. \tag{41}$$

Hence if ω_0 and μ_0 are a pair of corresponding values of ω and μ on the characteristic, it follows from (40) that

$$\omega - \omega_0 = f(\mu_0) - f(\mu) \quad \text{(along a } \xi \text{ char.)}, \tag{42}$$

$$\omega - \omega_0 = f(\mu) - f(\mu_0) \quad \text{(along a } \zeta \text{ char.)}. \tag{43}$$

We see from these relations that if ω is known at every point of a ξ or ζ characteristic and if μ is known at one point, then μ is determined at every point of the characteristic.

We now consider the special case of irrotational flow for which the ξ characteristics are straight lines and the inclination ω as well as

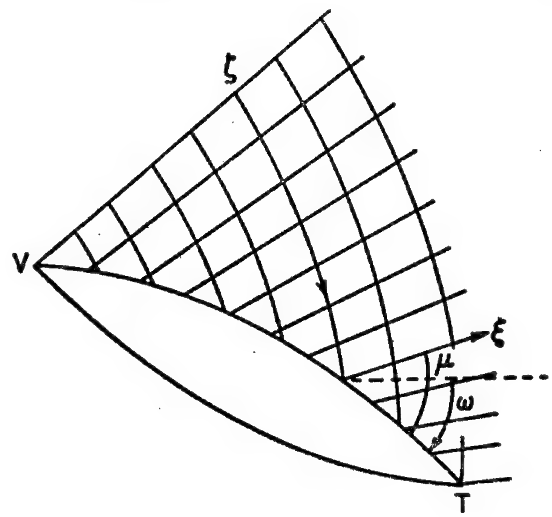


FIG. 6

the various hydrodynamical quantities are constant along these characteristics in accordance with *Remark* 2 of §6. Fig. 6 shows the family of straight line ξ characteristics issuing from the points of the profile (upper contour) which is of course one of the stream lines in the field of flow. The derivative in the left member of (33) is now indeterminate; however, it may be supposed that (34) and consequently (43) are valid relations. We assume that the ξ characteristic through the vertex V is intersected by every ζ characteristic which passes through a point of the profile between V and T as shown in Fig. 6. But along the ξ characteristic through V the angle ω and the inclination μ are constant and have the values ω_v and μ_v of these quantities at the vertex V. Hence the equation

$$\omega - \omega_v = f(\mu) - f(\mu_v) \tag{44}$$

holds along each ζ characteristic, and hence in particular this equation is satisfied along the profile itself. Correspondingly if it is assumed that the ζ characteristics form a family of straight lines we deduce the validity, along the profile, of the equation

$$\omega - \omega_v = f(\mu_v) - f(\mu). \tag{45}$$

Equation (44) is basic in the standard theory of pressure calculation along the profile [3]. The flow associated with this relation is *expansive*, i.e., the inclination μ of the characteristic lines relative to the profile decreases as ω decreases, or, in other words, as we move from the front vertex V to the tail vertex T of the profile. For it follows immediately from (34) that v increases along the profile as ω decreases, hence from (35) c decreases and hence $m = v/c$ increases. But from the relation $\sin \mu = 1/m$ it therefore follows that μ decreases as stated. Solving the relations $c^2 = \gamma p/\rho$ and $p = N\rho^\gamma$ for p and ρ we find

$$p = \frac{c^{2\gamma/\gamma-1}}{\gamma^{\gamma/\gamma-1}N^{1/\gamma-1}}, \qquad \rho = \frac{c^{2/\gamma-1}}{(\gamma N)^{1/\gamma-1}}.$$

Since c decreases as ω decreases, it follows from these equations that p and ρ likewise decrease as ω decreases in the case of expansive flow.

When the flow is governed by equation (45) it is clear on the basis of the above discussion that μ will *increase* as ω decreases so that the ξ characteristics, which are now curved lines, will tend to crowd together as we move along the profile in the direction of the flow. It will be seen in the following that a condition of this type will play an important role in the process of separation of the flow from the profile.

9. Standard pressure calculation along the profile. Consider the equations $p = N\rho^\gamma$ and $c^2 = \gamma p/\rho$ along the profile and also the corresponding equations $p_v = N\rho_v^\gamma$ and $c_v^2 = \gamma p_v/\rho_v$ where the subscript denotes evaluation at the vertex V of the profile. From these relations we can readily deduce

$$\frac{p}{p_v} = \left(\frac{\rho}{\rho_v}\right)^\gamma = \left(\frac{c^2}{c_v^2}\right)^{\frac{\gamma}{\gamma-1}}. \tag{46}$$

Also by elimination of v^2 between (35) and (36) we obtain an equation which can be written in the form

$$c^2 = 2(\gamma - 1)\, H \left(\frac{\sin^2\mu}{\gamma - \cos 2\mu}\right). \tag{47}$$

Hence from (46) and (47) we obtain

$$\frac{p}{p_v} = \frac{g(\mu)}{g(\mu_v)}, \tag{48}$$

where μ_v denotes the value of μ at the vertex V, and

$$g(\mu) = \left(\frac{\sin^2\mu}{\gamma - \cos 2\mu}\right)^{\frac{\gamma}{\gamma-1}}. \tag{49}$$

In the *standard* pressure calculation we seek the values of the ratio

$$\frac{p}{p_1} = \frac{p}{p_v} \cdot \frac{p_v}{p_1} \tag{50}$$

along the profile, where p_1 is the pressure of the undisturbed flow. This calculation is based on an irrotational flow of expansive type as explained at the end of the preceding section and is carried out as follows: First, we determine the values of μ_v and the ratio p_v/p_1 by means of the *shock conditions* (see §14). The value of μ at a point P of the profile can be found from the equation (44), or

$$f(\mu) = f(\mu_v) - \omega_v + \omega \tag{51}$$

in which ω is the known value of the inclination of the profile at P. Equation (48) can then be used to find p/p_v at the point P, after which equation (50) gives the required value of p/p_1 at P. Values of the corresponding ratio for the various hydrodynamical quantities, for example the ratio ρ/ρ_1 where ρ_1 is the density of the undisturbed flow, can be determined along the profile from the values of the pressure ratio p/p_1 by means of formulas which are readily constructed (see §10).

The above calculation of the pressure ratio p/p_1 along a profile is greatly facilitated by the use of a table, such as Table 2, for the functions $f(\mu)$ and $g(\mu)$. This table, which is based on the value of $\gamma = 1.4$, has been used in the pressure calculations for the $GU2$ and $GU3$ profiles in the latter part of this article.

10. Auxiliary formulas. Let us suppose that the quantity μ and the pressure ratio p/p_1 have been calculated along a profile by the method of the preceding section. From this calculation the various hydrodynamical quantities will be determined along the profile and the known algebraic relations between them will be satisfied. We shall now give a direct demonstration of these facts after which we shall derive certain formulas of differentiation which will be needed in the following section.

TABLE 2

μ	$f(\mu)$	$g(\mu)$	μ	$f(\mu)$	$g(\mu)$
90°	130.45°	.5283	44°	120.32°	.2971
89°	130.45	.5282	43°	119.54	.2860
88°	130.45	.5279	42°	118.71	.2746
87°	130.45	.5274	41°	117.83	.2630
86°	130.45	.5268	40°	116.90	.2512
85°	130.44	.5259	39°	115.92	.2391
84°	130.44	.5249	38°	114.88	.2269
83°	130.42	.5236	37°	113.77	.2146
82°	130.41	.5222	36°	112.61	.2022
81°	130.39	.5206	35°	111.37	.1897
80°	130.37	.5188	34°	110.07	.1772
79°	130.34	.5168	33°	108.70	.1647
78°	130.31	.5146	32°	107.23	.1522
77°	130.27	.5122	31°	105.69	.1399
76°	130.22	.5096	30°	104.08	.1278
75°	130.16	.5067	29°	102.36	.1159
74°	130.10	.5037	28°	100.56	.1043
73°	130.03	.5005	27°	98.65	.09313
72°	129.95	.4970	26°	96.64	.08237
71°	129.85	.4933	25°	94.53	.07211
70°	129.75	.4894	24°	92.30	.06243
69°	129.64	.4853	23°	89.96	.05338
68°	129.51	.4809	22°	87.50	.04502
67°	129.37	.4763	21°	84.91	.03739
66°	129.22	.4715	20°	82.19	.03054
65°	129.05	.4664	19°	79.34	.02447
64°	128.87	.4611	18°	76.34	.01919
63°	128.67	.4555	17°	73.21	.01469
62°	128.45	.4497	16°	69.94	.01095
61°	128.22	.4436	15°	66.51	.00791
60°	127.97	.4372	14°	62.95	.00552
59°	127.69	.4306	13°	59.23	.00370
58°	127.40	.4237	12°	55.37	.00237
57°	127.08	.4165	11°	51.37	.00143
56°	126.74	.4090	10°	47.22	.00081
55°	128.38	.4013	9°	42.94	.00043
54°	125.99	.3933	8°	38.53	.00020
53°	125.57	.3850	7°	34.00	.00009
52°	125.12	.3763	6°	29.36	.00003
51°	124.64	.3675	5°	24.63	.00001
50°	124.14	.3583	4°	19.81	.00000
49°	123.59	.3488	3°	14.92	.00000
48°	123.02	.3390	2°	9.98	.00000
47°	122.40	.3289	1°	4.00	.00000
46°	121.75	.3186	0°	0.00	.00000
45°	121.06	.3080			

The relation $p = N\rho^\gamma$, which must hold along the profile with N constant, becomes $p_v = N\rho_v^\gamma$ when evaluated at the vertex V. Combining the latter with the corresponding relation $p_1 = N_1\rho_1^\gamma$ for the undisturbed flow we have

$$\frac{N}{N_1} = \frac{p_v/p_1}{(\rho_v/\rho_1)^\gamma}. \tag{52}$$

This equation determines the constant N/N_1 since the ratios p_v/p_1 and ρ_v/ρ_1 are determined from the shock conditions (§14). Then from $p = N\rho^\gamma$ and $p_1 = N_1\rho_1^\gamma$ we obtain

$$\frac{\rho}{\rho_1} = \left(\frac{p/p_1}{N/N_1}\right)^{1/\gamma} \tag{53}$$

for the determination of the density ratio ρ/ρ_1 along the profile. Similarly from $c^2 = \gamma p/\rho$ and $c_1^2 = \gamma p_1/\rho_1$ we have

$$\frac{c^2}{c_1^2} = \frac{p/p_1}{\rho/\rho_1} \tag{54}$$

by which the ratio c^2/c_1^2 is given along the profile. Now from $\sin\mu = 1/m$ we can determine the mach number m, since μ is known by direct calculation. Hence

$$m^2 = \frac{v^2}{c^2} = \frac{(v/W)^2W^2}{(c^2/c_1^2)c_1^2} = \frac{M^2(v/W)^2}{c^2/c_1^2},$$

where W has been used to denote the velocity and M the mach number in the undisturbed flow. But this equation gives

$$\left(\frac{v}{W}\right)^2 = \frac{m^2(c^2/c_1^2)}{M^2} \tag{55}$$

for the determination of the ratio v/W along the profile.

If we assume that the values p_1, ρ_1, W, c_1 and M are known in the undisturbed flow, then the determination of the ratio p/p_1 and the above ratios (53), (54) and (55) yields the value of the quantities p, ρ, v and c along the profile. The constant N is determined from (52). Moreover these quantities and m satisfy the relations $m^2 = v^2/c^2$, $c^2 = \gamma p/\rho$ and $p = N\rho^\gamma$ along the profile.

Let us now show that the energy equation (3) is satisfied along the profile. Thus

$$\begin{aligned}\frac{\gamma p}{(\gamma-1)\rho} + \frac{v^2}{2} &= \frac{\gamma p}{(\gamma-1)\rho} + \frac{m^2c^2}{2} \\ &= \frac{\gamma p}{\rho}\left(\frac{1}{\gamma-1} + \frac{m^2}{2}\right)\end{aligned}$$

$$= \frac{\gamma(p/p_v)p_v}{(\rho/\rho_v)\rho_v}\left(\frac{1}{\gamma - 1} + \frac{1}{2\sin^2\mu}\right).$$

But p/p_v is given by (48) and this ratio is related to the ratio ρ/ρ_v by the equation $p/p_v = (\rho/\rho_v)^\gamma$. Hence

$$\begin{aligned}\frac{\gamma p}{(\gamma - 1)\rho} + \frac{v^2}{2} &= \frac{\gamma p_v}{\rho_v}\left(\frac{1}{\gamma - 1} + \frac{1}{2\sin^2\mu}\right)\left[\frac{g(\mu)}{g(\mu_v)}\right]^{(\gamma-1)/\gamma} \\ &= c_v^2\left[\frac{\gamma - 1 + 2\sin^2\mu}{2(\gamma - 1)\sin^2\mu}\right]\left(\frac{\sin^2\mu}{\gamma - \cos 2\mu}\right)\left(\frac{\gamma - \cos 2\mu_v}{\sin^2\mu_v}\right) \\ &= \frac{c_v^2(\gamma - \cos 2\mu_v)}{2(\gamma - 1)\sin^2\mu_v} = \frac{c_v^2(\gamma - 1 + 2\sin^2\mu_v)}{2(\gamma - 1)\sin^2\mu_v} \\ &= \frac{c_v^2}{\gamma - 1} + \frac{c_v^2 m_v^2}{2} = \frac{\gamma p_v}{(\gamma - 1)\rho_v} + \frac{v_v^2}{2}.\end{aligned}$$

But this last expression has the value H by the shock conditions (see *Remark* 1 in §2). Hence the energy equation holds along the profile. For later use we also note the validity, along the profile, of the relation (36) or (37), which is an immediate consequence of the energy equation.

Let us now differentiate (48) along the profile with respect to the arc length σ to obtain

$$\frac{dp}{d\sigma} = p_v \frac{g'(\mu)}{g(\mu_v)} \frac{d\mu}{d\sigma}. \tag{56}$$

Differentiating (49) with respect to σ we find, after some reduction, that

$$g'(\mu) = \frac{2\gamma g(\mu) \operatorname{ctn} \mu}{\gamma - \cos 2\mu}. \tag{57}$$

Also differentiation of (37) gives

$$\frac{d\mu}{d\sigma} = \frac{-2(\gamma - 1)H}{v^3 \sin 2\mu} \frac{dv}{d\sigma}. \tag{58}$$

When we substitute (57) and (58) into (56) and use (36) and (48), the resulting equation can be written

$$\frac{dp}{d\sigma} = \frac{-\gamma p}{v \sin^2 \mu} \frac{dv}{d\sigma}. \tag{59}$$

For definiteness we shall now suppose that the flow is of the expansive type (§8) so that (44) holds along the profile. Then differentiating this equation with respect to σ we have

$$\frac{d\omega}{d\sigma} = f'(\mu)\,\frac{d\mu}{d\sigma}. \tag{60}$$

But from (41) we find, after some reduction, that

$$f'(\mu) = \frac{2}{\gamma \sec^2\mu + \tan^2\mu - 1}. \tag{61}$$

Hence substituting (58) and (61) into (60) an equation is obtained which reduces to

$$\frac{d\omega}{d\sigma} = \frac{-\operatorname{ctn}\mu}{v}\,\frac{dv}{d\sigma}. \tag{62}$$

We shall also need formulas for the normal derivatives $d\omega/dn$ and dp/dn along the profile. In Fig. 7 we have shown two ξ characteristics

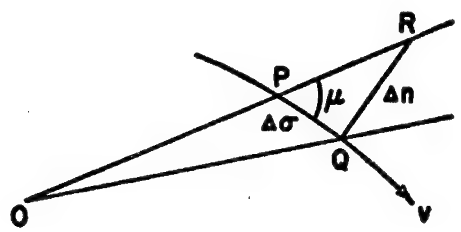


FIG. 7

which pass through points P and Q of the profile and intersect at a point O. QR is a normal to the profile at point Q. Denoting the values of ω at points P, Q, R by ω_P, ω_Q, ω_R, respectively, the change $\Delta\omega$, incurred in moving along the normal from Q to R, is $\omega_R - \omega_Q$, and hence is also equal to $\omega_P - \omega_Q$ since ω is constant on each ξ characteristic for the expansive flow under consideration. But to within terms of the first order we have

$$\omega_P - \omega_Q = -\frac{d\omega}{d\sigma}\,\Delta\sigma.$$

Hence dividing both members of this relation by $\Delta n = QR$ and passing to the limit as $Q \to P$ we obtain

$$\frac{d\omega}{dn} = -\operatorname{ctn}\mu\,\frac{d\omega}{d\sigma} = \frac{\operatorname{ctn}^2\mu}{v}\,\frac{dv}{d\sigma} \tag{63}$$

when use is made of (62). In a similar manner, using (59), we are led to the equation

$$\frac{dp}{dn} = -\operatorname{ctn}\mu\,\frac{dp}{d\sigma} = \frac{\gamma p \operatorname{ctn}\mu}{v \sin^2 \mu}\,\frac{dv}{d\sigma}. \tag{64}$$

11. Proof of sufficiency. By the pressure calculation of §9 for expansive flow the inclination μ of the ξ characteristics relative to the profile is determined, and hence these characteristics are determined since they are straight lines. The quantities p, ρ, ω, v, u_α, c and m are constant along the ξ characteristics for this flow (§8). Hence let us determine their values along the profile as in §9 and §10 and then define them in the plane of the flow by the requirement that they be constant along the curves of the congruence of straight lines which make an angle μ with the profile. We now raise the question: Will the quantities p, ρ, ω, v, u_α, c and m, so determined in the plane, be the pressure, density, etc., of an irrotational flow of expansive type having the above congruence of straight lines as its ξ characteristics?

To give a direct proof of the fact that the above question is to be answered in the affirmative, we make use of the following result [2]: *Let p, ρ, v and ω be a set of functions of the coordinates, and denote by G the congruence of curves having the inclination ω. Suppose that these functions satisfy the conditions*

$$(m^2 - 1)\,\frac{dp}{d\sigma} = -\rho v^2\,\frac{d\omega}{dn}, \qquad \frac{dp}{dn} = -\rho v^2\,\frac{d\omega}{d\sigma} \tag{65}$$

where σ denotes arc length along the curves of G, and n is the distance along their normals defined as in §7 relative to the direction on these curves specified by the components

$$u_1 = v\cos\omega, \qquad u_2 = v\sin\omega. \tag{66}$$

Suppose, furthermore, that

$$\frac{\gamma}{\gamma - 1}\,\frac{p}{\rho} + \frac{v^2}{2} = H, \qquad p = N\rho^\gamma \tag{67}$$

where H is an absolute (positive) constant, and N is a function of the coordinates which is constant along the curves of the congruence G. Then the p, ρ and the u_α defined by (66) are the pressure, density and velocity components of a compressible flow having the curves of the congruence G as its stream lines. It is now immediately evident that (66) and (67) are satisfied in the plane, since these relations hold along the profile (§10) and all quantities involved in the relations are constant by hypothesis along

the straight lines of the above congruence. Moreover, (65) holds along the profile. For from (59) and (63) we have

$$(68) \qquad (m^2 - 1)\frac{dp}{d\sigma} = \frac{-\gamma p \operatorname{ctn}^2\mu}{v \sin^2\mu}\frac{dv}{d\sigma} = -\rho v \operatorname{ctn}^2\mu \frac{dv}{d\sigma},$$

$$(69) \qquad -\rho v^2 \frac{d\omega}{dn} = -\rho v \operatorname{ctn}^2 \mu \frac{dv}{d\sigma},$$

along the profile, where we have used the relations $m = v/c$ and $\sin \mu = 1/m$ in the derivation of (68). Hence from (68) and (69) we obtain the first equation (65). Similarly from (62) and (64) we have

$$\frac{dp}{dn} = \rho v \operatorname{ctn} \mu \frac{dv}{d\sigma},$$

$$-\rho v^2 \frac{d\omega}{d\sigma} = \rho v \operatorname{ctn} \mu \frac{dv}{d\sigma},$$

and hence the second equation (65) is also satisfied along the profile.

Now consider Fig. 8 which is identical with Fig. 7 except that in Fig. 8 we have indicated a second stream line along which the velocity is

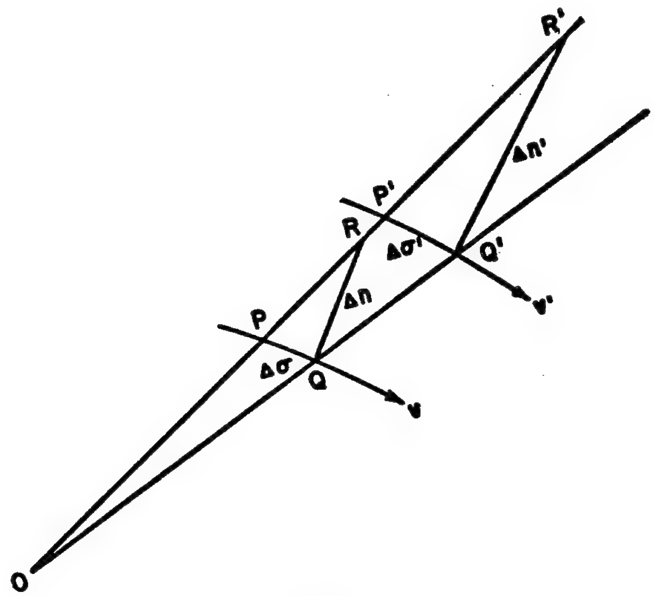


Fig. 8

designated by v' and arc length by σ'. Also $Q'R'$ is normal to the second stream line at Q'. Let f be a function which is constant along the curves

of the straight line congruence. Hence we have

$$f_Q - f_P = f_{Q'} - f_{P'} \quad \text{(by hypothesis)},$$

$$\frac{PQ}{OP} = \frac{P'Q'}{OP'} \quad \text{(approximately)}.$$

Dividing corresponding members of these equations and taking the limit as $Q \to P$ we see that

$$\left(\frac{df}{d\sigma}\right)_P = \frac{OP'}{OP}\left(\frac{df}{d\sigma}\right)_{P'}. \tag{70}$$

Similarly, if F is constant along the curves of the straight line congruence, we have

$$F_R - F_Q = F_{R'} - F_{Q'} \quad \text{(by hypothesis)},$$

$$\frac{QR}{OQ} = \frac{Q'R'}{OQ'} \quad \text{(exactly)}.$$

Hence dividing corresponding members of these equations and letting $Q \to P$ we obtain

$$\left(\frac{dF}{dn}\right)_P = \frac{OP'}{OP}\left(\frac{dF}{dn}\right)_{P'}. \tag{71}$$

Now identify f with p and F with ω in the first equation (65). Then when we make the substitutions (70) and (71) in the left and right members of the first equation (65) we see that the factor OP'/OP cancels, and we deduce the validity of this equation at the point P'. In the same way by taking $F = p$ and $f = \omega$ it follows that the second equation (65) holds at P'. Hence (65) holds in the plane of the flow which is traversed by the straight line congruence, since P' is any point in this portion of the plane.

It remains to mention that the flow determined in the plane by the above procedure is *irrotational;* this follows from §6 since N in equation (67) is an absolute constant by our hypothesis, i.e., the flow is isentropic. Moreover, the flow will have the profile as one of its stream lines and will evidently be of expansive type with the above congruence of straight lines as its ξ characteristics, since μ is an increasing function of the inclination ω of the profile by direct calculation (§9). We have now proved the following result: *If the inclination μ and the various hydrodynamical quantities are calculated along the profile by the procedure of §9 and §10, and if the values of these quantities are extended into the plane by the requirement that they be constant along the lines of the congruence of straight lines which make an angle μ with the profile, then there will be*

defined in the plane an irrotational flow of expansive type having the profile as a stream line and the straight lines of the above congruence as its ξ characteristics. It is evident that a corresponding result will hold when the equation (45) lies at the basis of the calculation, in which case the ζ characteristics will replace the ξ characteristics in the discussion.

12. The continuity assumption. While the discussion and results of the preceding sections are basic in this theory of the separation of supersonic flow from curved profiles, these results will not in themselves account for the phenomenon of separation. This is done by the assumptions appearing in this and the next section. The first of these assumptions is the following:

Continuity assumption. The pressure, density and velocity along the profile are continuous functions of the arc length. As a matter of fact the condition that any one of these functions be continuous would imply the continuity of the others on account of the relations between the hydrodynamical quantities in §10.

It will be found that, in consequence of the above assumption and the assumptions of §13, the pressure calculation along the profile must be carried out in part on the basis of the relation (44) for expansive flow and in part on the basis of (45) for flow of non-expansive type or *compressive* flow. This gives rise to the possibility of a shock, separating the regions corresponding to these two types of flow, of which there is some indication in the photographs [1]. Any such shock cannot extend to a point on the profile in the case of an actual viscous fluid since the velocity will not be supersonic in a sufficiently small region about the profile, and this circumstance appears in close agreement with the above assumption of continuity along the profile. It must be borne in mind, moreover, that the simple irrotational flow of expansive or compressive type, discussed in the preceding sections, cannot strictly exist in the region about the profile and that our pressure calculations along the profile, based on this simple model, are intended only as approximations. In fact, from the curvature of the profile we can deduce as a mathematical consequence the curvature of the shock line which is associated with the front vertex V of the profile as well as the rotational character of the flow in the region behind this shock line.

13. Separation assumptions. The separation of supersonic flow from a curved profile has its origin in the conditions existing at the rear of the profile. These conditions are effectively contained in the following italicized assumptions, but before stating them it will be helpful to make certain preliminary remarks.

It will be assumed that behind the front vertex V of the profile the pressure ratio p/p_1 (as well as the corresponding ratios ρ/ρ_1, etc.) can be calculated, to the extent permitted by the following separation assumptions, as for a flow of expansive type (§9 and §10). After separation effects occur the ratio p/p_1 so calculated does not retain its physical validity and will consequently be designated by the term *virtual*. Similarly we may speak of the *virtual density ratio* ρ/ρ_1, the *virtual mach angle* μ, etc., the angle μ in the above discussion being commonly called the mach angle. These virtual magnitudes will enable us to express, in a certain sense, rear conditions of significance in the hydrodynamical problem.

We have spoken in the introduction (§1) of the tendency of certain mach angles to remain invariant during the separation process. This is done in part by the following assumption (γ), in accordance with which the configuration consisting of the virtual ξ characteristics and the tangent to the profile at the point of separation is transformed into the configuration consisting of the rear shock line and the line of flow behind the rear shock at the vertex of this shock. A certain rigidity is therefore imposed on the flow pattern after separation by the assumption (γ). Account is taken of conditions at the rear of the profile by a corresponding tendency toward invariance of the virtual mach angle μ at the tail vertex of the profile and the mach angle of the undisturbed flow, defined by $\sin^{-1} 1/M$, which is obviously of significance for the flow in the region behind the profile. The effect of these two factors on the value of the actual mach angle μ at the point of separation is continued in assumption (δ). Other assumptions made appear to be self-explanatory and need not be commented upon separately. The complete set of separation assumptions is as follows:

(α) *When separation occurs the flow leaves the profile along the tangent at the point of separation.*

(β) *The pressure is constant along the tangent stream line at the point of separation and the pressure on the profile behind the separation point is very approximately equal to this constant pressure.*

(γ) *After separation there occurs at the rear of the profile a shock possessing the following properties:* (*a*) *the flow behind the shock is parallel to the undisturbed flow, and* (*b*) *the direction of the flow behind the shock forms with the shock line at its vertex an angle which is equal to the virtual mach angle at the point of separation.*

(δ) *In the actual flow the mach angle* μ *at the point of separation is equal to the arithmetic mean of the mach angle for the undisturbed flow and the virtual mach angle* μ *at the tail of the profile.*

Naturally the above assumptions are intended only as quantitative approximations to conditions existing in the actual flow. In the following sections we shall apply this theory to the $GU2$ and $GU3$ profiles for which experimental data is available. *It will be found that the separation effects obtained when considered in themselves, i.e., apart from errors due to the standard calculation of* §9 *on which these effects also depend, are much better quantitatively than results based on the standard calculation along that part of the profile where such calculations apply* (*see* §20).

14. Application of the shock conditions. Consider the following two shock conditions [4]

$$\frac{p}{p_1} = \frac{2\gamma}{\gamma + 1}(M^2 \sin^2 \alpha - 1) + 1, \tag{72}$$

$$\frac{\rho}{\rho_1} = \frac{2(M^2 \sin^2 \alpha - 1)}{(\gamma - 1)M^2 \sin^2 \alpha + 2} + 1, \tag{73}$$

where M, p_1, and ρ_1 are the mach number, pressure and density of the undisturbed stream, while p and ρ denote pressure and density immediately behind the shock line. The inclination of the shock line, relative to the x^1 axis, which is assumed to have the direction of the undisturbed flow, is given by α and is related to the inclination ω of the stream line, also relative to the x^1 axis, immediately behind the shock line by the equation

$$\tan \omega = \frac{2[(M^2 - 1) \tan^2 \alpha - 1]}{[(\gamma - 1)M^2 + 2] \tan^3 \alpha + [(\gamma + 1)M^2 + 2] \tan \alpha}. \tag{74}$$

If W is the velocity of the undisturbed stream and v denotes velocity immediately behind the shock line, we can derive an expression for the ratio v/W as follows. We have

$$\frac{\gamma p}{(\gamma - 1)\rho} + \frac{v^2}{2} = \frac{\gamma p_1}{(\gamma - 1)\rho_1} + \frac{W^2}{2} = \frac{C^2}{(\gamma - 1)} + \frac{W^2}{2}$$

from the invariance of the left member of (3) in the passage across the shock line (see *Remark* 1 in §2). In the above relation C, given by $C^2 = \gamma p_1/\rho_1$, is the velocity of sound in the undisturbed stream. Hence

$$\frac{\gamma(p/p_1)p_1}{(\gamma - 1)(\rho/\rho_1)\rho_1} + \frac{(v/W)^2 W^2}{2} = \frac{C^2}{(\gamma - 1)} + \frac{W^2}{2}, \quad \text{or}$$

$$\frac{p/p_1}{(\gamma - 1)M^2(\rho/\rho_1)} + \frac{(v/W)^2}{2} = \frac{1}{(\gamma - 1)M^2} + \frac{1}{2}, \quad \text{or}$$

$$\left(\frac{v}{W}\right)^2 = \frac{(\gamma - 1)M^2 + 2}{(\gamma - 1)M^2} - \frac{2(p/p_1)}{(\gamma - 1)M^2(\rho/\rho_1)}. \tag{75}$$

The equations used in the calculation of α and the ratios p/p_1, ρ/ρ_1 and v/W are obtained from the above equations (74), (72), (73) and (75) by putting $\gamma = 1.4$. Making this substitution the equations become

$$\tan \omega = \frac{5[(M^2 - 1)\tan^2 \alpha - 1]}{(M^2 + 5)\tan^3 \alpha + (6M^2 + 5)\tan \alpha}, \tag{76}$$

$$\frac{p}{p_1} = \frac{7}{6}(M^2 \sin^2 \alpha - 1) + 1, \tag{77}$$

$$\frac{\rho}{\rho_1} = \frac{5(M^2 \sin^2 \alpha - 1)}{M^2 \sin^2 \alpha + 5}, \tag{78}$$

$$\left(\frac{v}{W}\right)^2 = \frac{M^2 + 5}{M^2} - \frac{5(p/p_1)}{M^2(\rho/\rho_1)}. \tag{79}$$

In addition, we will make use of an equation giving the mach number m immediately behind the shock line which is derived as follows. By definition

$$m^2 = \frac{v^2}{c^2} = \frac{v^2}{W_2}\frac{W^2}{c^2} = \left(\frac{v}{W}\right)^2 \frac{W^2}{\gamma p/\rho},$$

where $c^2 = \gamma p/\rho$ gives the velocity of sound c behind the shock line. Hence

$$m^2 = \left(\frac{v}{W}\right)^2 \frac{W^2}{\left(\frac{\gamma p_1}{\rho_1}\right)} \frac{\rho/\rho_1}{p/p_1} = \left(\frac{v}{W}\right)^2 \frac{\rho/\rho_1}{p/p_1}\frac{W^2}{C^2}$$

from which we obtain the desired formula

$$m^2 = M^2 \frac{(v/W)^2(\rho/\rho_1)}{p/p_1}. \tag{80}$$

After m has been determined, the equation $\sin \mu = 1/m$ can be used to determine the mach angle μ (§4).

Denote by ω_v the inclination of the profile at its vertex V and by a the angle of attack. We then easily calculate

$$\omega_v = 11.537° \quad \text{for} \quad GU2,\ a = 0,$$

$$\omega_v = 20.027° \quad \text{for} \quad GU3,\ a = 0.$$

Using these basic values of ω_v, the inclination ω_v is immediately determined for any angle of attack a for the $GU2$ and $GU3$ profiles. In Tables 3, 4 and 5 we have given the initial determinations, or determinations at the vertex V, of the above quantities for the $GU2$ and $GU3$ profiles at various angles of attack when $M = 2.13$ and $M = 1.85$.

TABLE 3

Initial determinations for GU2, M = 2.13

a(deg)	ω(deg)	α(deg)	p/p_1	ρ/ρ_1	$(v/W)^2$	m	μ(deg)
0	11.537	38.574	1.89118	1.56467	.77002	1.70010	36.03
4	7.537	34.472	1.52899	1.35130	.85508	1.85165	32.69
6	5.537	32.616	1.37108	1.25171	.89491	1.92526	31.29
8	3.537	30.863	1.22622	1.15652	.93359	1.99872	30.02
10	1.537	29.201	1.09319	1.06569	.97156	2.07292	28.84

TABLE 4

Initial determinations for GU3, M = 2.13

a(deg)	ω(deg)	α(deg)	p/p_1	ρ/ρ_1	$(v/W)^2$	m	μ(deg)
0	20.027	49.734	2.91520	2.07412	.55310	1.33617	48.45
5	15.027	42.638	2.26190	1.76369	.68868	1.56086	39.84
10	10.027	36.965	1.74727	1.48228	.80298	1.75799	34.67
14	6.027	33.071	1.40949	1.27628	.88504	1.90684	31.63

TABLE 5

Initial determinations for GU3, M = 1.85

a(deg)	ω(deg)	α(deg)	p/p_1	ρ/ρ_1	$(v/W)^2$	m	μ(deg)
4	16.027	51.286	2.26435	1.76494	.58662	1.25096	53.07
16	4.027	36.350	1.23609	1.16314	.90836	1.71038	35.78

Using the values of p/p_1 and μ from Tables 3, 4 and 5, we can now calculate the value of the pressure ratio p/p_1 at a point on the profile of arbitrary inclination ω by the method of §9. In addition to the pressure ratio p/p_1, the following tabulations[7] give, for the indicated values of ω, the determinations of the mach angle μ and the quantities $f(\mu)$, $g(\mu)$ and p/p_v which enter in the calculations as well as the determinations of a new quantity $\bar{m}$, defined in §16. The final value of ω in each of these tables gives the inclination of the tail vertex T for the angle of attack under consideration and the corresponding value of μ is the value of the mach angle μ_T at the tail vertex which we shall later have to consider in connection with the above assumption (δ).

[7] In these tables and in all following calculations we shall measure angles in degrees; with this understanding the degree designation will henceforth be omitted.

GU2, $M = 2.13$, $a = 0$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
11	112.10	35.59	.1971	.9729	1.8399	
9	110.10	34.02	.1774	.8756	1.6559	
7	108.10	32.59	.1596	.7878	1.4899	
5	106.10	31.27	.1432	.7068	1.3367	
3	104.10	30.01	.1279	.6313	1.1939	
1	102.10	28.86	.1143	.5642	1.0670	
−1	100.10	27.76	.1016	.5015	.9484	4.54
−3	98.10	26.73	.09022	.4453	.8421	4.74
−5	96.10	25.74	.07970	.3934	.7440	4.95
−7	94.10	24.81	.07027	.3468	.6559	5.18
−9	92.10	23.91	.06162	.3041	.5751	5.45
−11	90.10	23.06	.05392	.2661	.5032	5.74
−11.537	89.56	22.84	.05204	.2569	.4858	5.82

GU2, $M = 2.13$, $a = 4$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
7	107.70	32.32	.1562	.9714	1.4852	
6	106.70	31.66	.1480	.9204	1.4073	
5	105.70	31.01	.1400	.8707	1.3312	
4	104.70	30.39	.1325	.8240	1.2599	
3	103.70	29.78	.1252	.7786	1.1905	
2	102.70	29.20	.1183	.7357	1.1249	
1	101.70	28.63	.1116	.6940	1.0612	
0	100.70	28.08	.1052	.6542	1.0003	4.51
−1	99.70	27.55	.09927	.6174	.9439	4.60
−2	98.70	27.03	.09347	.5813	.8888	4.70
−3	97.70	26.53	.08807	.5477	.8374	4.80
−4	96.70	26.03	.08269	.5142	.7863	4.91
−5	95.70	25.55	.07775	.4835	.7393	5.01
−6	94.70	25.08	.07293	.4535	.6935	5.13
−7	93.70	24.63	.06853	.4262	.6516	5.24
−8	92.70	24.18	.06417	.3991	.6102	5.37
−9	91.70	23.74	.06008	.3736	.5713	5.51
−10	90.70	23.32	.05628	.3500	.5352	5.65
−11	89.70	22.89	.05246	.3262	.4988	5.80
−12	88.70	22.49	.04912	.3055	.4671	5.96
−13	87.70	22.08	.04569	.2841	.4345	6.14
−14	86.70	21.69	.04265	.2652	.4056	6.33
−15	85.70	21.31	.03976	.2473	.3781	6.53
−15.537	85.16	21.10	.03815	.2373	.3628	6.65

$GU2, M = 2.13, a = 6$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
5	105.60	30.94	.1392	.9700	1.3300	
4	104.60	30.32	.1317	.9178	1.2583	
3	103.60	29.72	.1245	.8676	1.1896	
2	102.60	29.14	.1176	.8195	1.1236	
1	101.60	28.58	.1110	.7735	1.0606	
0	100.60	28.02	.1045	.7282	.9984	4.53
−1	99.60	27.50	.09871	.6879	.9412	4.62
−2	98.60	26.98	.09291	.6475	.8877	4.72
−3	97.60	26.48	.08753	.6100	.8363	4.81
−4	96.60	25.98	.08216	.5725	.7850	4.92
−5	95.60	25.51	.07734	.5390	.7389	5.03
−6	94.60	25.03	.07242	.5047	.6919	5.14
−7	93.60	24.58	.06804	.4742	.6501	5.26
−8	92.60	24.13	.06369	.4438	.6085	5.39
−9	91.60	23.70	.05972	.4162	.5706	5.52
−10	90.60	23.27	.05582	.3890	.5333	5.67
−11	89.60	22.85	.05213	.3633	.4981	5.82
−12	88.60	22.45	.04878	.3399	.4661	5.98
−13	87.60	22.04	.04535	.3160	.4333	6.16
−14	86.60	21.65	.04235	.2951	.4046	6.34
−15	85.60	21.27	.03945	.2749	.3769	6.54
−16	84.60	20.89	.03664	.2553	.3501	6.76
−17	83.60	20.52	.03410	.2376	.3258	7.00
−17.537	83.06	20.32	.03273	.2281	.3127	7.17

$GU2, M = 2.13, a = 8$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
3	103.58	29.71	.1243	.9703	1.1899	
2	102.58	29.13	.1174	.9165	1.1238	
1	101.58	28.57	.1109	.8657	1.0616	
0	100.58	28.01	.1044	.8150	.9994	4.53
−1	99.58	27.49	.09860	.7697	.9438	4.63
−2	98.58	26.97	.09281	.7245	.8884	4.72
−3	97.58	26.47	.08743	.6825	.8369	4.82
−4	96.58	25.97	.08206	.6406	.7855	4.92
−5	95.58	25.50	.07724	.6030	.7394	5.03
−6	94.58	25.02	.07232	.5646	.6923	5.15
−7	93.58	24.57	.06795	.5304	.6504	5.27
−8	92.58	24.13	.06369	.4972	.6097	5.39
−9	91.58	23.69	.05962	.4654	.5707	5.52
−10	90.58	23.26	.05573	.4351	.5335	5.67
−11	89.58	22.85	.05213	.4070	.4990	5.82
−12	88.58	22.44	.04870	.3802	.4662	5.98
−13	87.58	22.03	.04527	.3534	.4333	6.16

$GU2$, $M = 2.13$, $a = 8$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
−14	86.58	21.64	.04227	.3300	.4046	6.34
−15	85.58	21.26	.03937	.3073	.3769	6.55
−16	84.58	20.88	.03657	.2855	.3501	6.77
−17	83.58	20.51	.03403	.2657	.3257	7.01
−18	82.58	20.14	.03150	.2459	.3015	7.27
−19	81.58	19.79	.02927	.2285	.2802	7.55
−19.537	81.04	19.60	.02811	.2194	.2691	7.72

$GU2$, $M = 2.13$, $a = 10$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
1	101.53	28.54	.1106	.9702	1.0606	
0	100.53	27.98	.1041	.9132	.9983	4.54
−1	99.53	27.46	.09827	.8620	.9424	4.63
−2	98.53	26.94	.09248	.8112	.8868	4.73
−3	97.53	26.44	.08710	.7640	.8352	4.83
−4	96.53	25.95	.08186	.7181	.7850	4.93
−5	95.53	25.47	.07693	.6748	.7377	5.04
−6	94.53	25.00	.07211	.6325	.6915	5.16
−7	93.53	24.55	.06775	.5943	.6499	5.27
−8	92.53	24.10	.06340	.5561	.6080	5.40
−9	91.53	23.67	.05944	.5214	.5700	5.54
−10	90.53	23.24	.05555	.4873	.5327	5.67
−11	89.53	22.83	.05196	.4558	.4983	5.83
−12	88.53	22.42	.04853	.4257	.4654	5.99
−13	87.53	22.01	.04510	.3956	.4325	6.17
−14	86.53	21.63	.04220	.3702	.4047	6.35
−15	85.53	21.24	.03922	.3440	.3761	6.56
−16	84.53	20.86	.03643	.3196	.3493	6.78
−17	83.53	20.49	.03390	.2974	.3251	7.02
−18	82.53	20.12	.03136	.2751	.3007	7.28
−19	81.53	19.77	.02914	.2556	.2794	7.56
−20	80.53	19.42	.02702	.2370	.2591	7.88
−21	79.53	19.07	.02489	.2183	.2387	8.23
−21.537	78.99	18.88	.02384	.2091	.2286	8.44

$GU3,\ M = 2.13,\ a = 0$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
20	123.25	48.40	.3429	.9985	2.9108	
17	120.25	43.91	.2961	.8623	2.5138	
14	117.25	40.38	.2557	.7446	2.1707	
11	114.25	37.43	.2199	.6404	1.8669	
8	111.25	34.91	.1886	.5492	1.6010	
5	108.25	32.69	.1608	.4683	1.3652	
2	105.25	30.73	.1366	.3978	1.1597	
−1	102.25	28.94	.1152	.3355	.9781	4.22
−4	99.25	27.31	.09659	.2813	.8200	4.52
−7	96.25	25.82	.08052	.2345	.6826	4.86
−10	93.25	24.43	.06659	.1939	.5653	5.26
−13	90.25	23.12	.05447	.1586	.4624	5.73
−16	87.25	21.90	.04426	.1289	.3758	6.31
−19	84.25	20.76	.03575	.1041	.3035	7.05
−20.027	83.22	20.38	.03314	.09651	.2813	7.35

$GU3,\ M = 2.13,\ a = 5$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
15	116.71	39.81	.2489	.9984	2.2583	
12	113.71	36.95	.2140	.8584	1.9416	
9	110.71	34.49	.1833	.7353	1.6632	
6	107.71	32.33	.1563	.6270	1.4182	
3	104.71	30.39	.1325	.5315	1.2022	
0	101.71	28.64	.1117	.4481	1.0136	4.35
−3	98.71	27.03	.09347	.3749	.8480	4.65
−6	95.71	25.56	.07786	.3123	.7064	4.97
−9	92.71	24.18	.06417	.2574	.5822	5.35
−12	89.71	22.90	.05254	.2108	.4768	5.80
−15	86.71	21.69	.04265	.1711	.3870	6.36
−18	83.71	20.56	.03438	.1379	.3119	7.06
−21	80.71	19.48	.02738	.1098	.2484	7.98
−24	77.71	18.46	.02162	.08672	.1962	9.23
−25.027	76.68	18.11	.01977	.07930	.1794	9.77

$GU3$, $M = 2.13$, $a = 10$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
10	110.91	34.65	.1853	.9984	1.7445	
7	107.91	32.46	.1580	.8513	1.4875	
4	104.91	30.52	.1341	.7225	1.2624	
1	101.91	28.75	.1130	.6088	1.0637	
−2	98.91	27.14	.09469	.5102	.8915	4.67
−5	95.91	25.65	.07878	.4245	.7417	4.98
−8	92.91	24.27	.06504	.3504	.6122	5.34
−11	89.91	22.98	.05321	.2867	.5009	5.77
−14	86.91	21.77	.04327	.2331	.4073	6.29
−17	83.91	20.63	.03486	.1878	.3281	6.95
−20	80.91	19.55	.02781	.1498	.2617	7.81
−23	77.91	18.52	.02194	.1182	.2065	8.96
−26	74.91	17.54	.01712	.09224	.1612	10.58
−29	71.91	16.60	.01319	.07107	.1242	13.04
−30.027	70.88	16.29	.01203	.06482	.1133	14.18

$GU3$, $M = 2.13$, $a = 14$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
6	106.63	31.61	.1474	.9986	1.4074	
3	103.63	29.74	.1247	.8446	1.1904	
0	100.63	28.04	.1048	.7100	1.0007	4.52
−3	97.63	26.49	.08764	.5938	.8369	4.81
−6	94.63	25.05	.07262	.4920	.6934	5.14
−9	91.63	23.71	.05981	.4052	.5711	5.52
−12	88.63	22.46	.04887	.3311	.4667	5.97
−15	85.63	21.28	.03953	.2678	.3774	6.54
−18	82.63	20.16	.03164	.2144	.3022	7.26
−21	79.63	19.10	.02508	.1699	.2395	8.23
−24	76.63	18.10	.01972	.1336	.1883	9.50
−27	73.63	17.13	.01528	.1035	.1459	11.38
−30	70.63	16.21	.01174	.07954	.1121	14.28
−33	67.63	15.33	.00891	.06037	.0851	19.31
−34.027	66.60	15.03	.00800	.05420	.0764	22.06

$GU3,\ M = 1.85,\ a = 4$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
16	125.57	53.00	.3850	.9984	2.2607	
13	122.57	47.27	.3316	.8600	1.9473	
10	119.57	43.04	.2864	.7427	1.6817	
7	116.57	39.66	.2471	.6408	1.4510	
4	113.57	36.83	.2105	.5459	1.2361	
1	110.57	34.38	.1820	.4720	1.0688	
−2	107.57	32.23	.1551	.4022	.9107	3.46
−5	104.57	30.30	.1314	.3408	.7717	3.79
−8	101.57	28.56	.1108	.2873	.6505	4.14
−11	98.57	26.96	.09271	.2404	.5443	4.54
−14	95.57	25.49	.07714	.2001	.4531	5.00
−17	92.57	24.12	.06359	.1649	.3734	5.57
−20	89.57	22.84	.05204	.1350	.3057	6.27
−23	86.57	21.64	.04227	.1096	.2482	7.18
−24.027	85.54	21.24	.03922	.1017	.2303	7.11

$GU3,\ M = 1.85,\ a = 16$

ω	$f(\mu)$	μ	$g(\mu)$	p/p_v	p/p_1	$\bar{m}^2$
4	112.31	35.76	.1992	.9985	1.2342	
1	109.31	33.45	.1703	.8536	1.0551	
−2	106.31	31.40	.1448	.7258	.8972	3.62
−5	103.31	29.55	.1224	.6135	.7583	3.95
−8	100.31	27.87	.1028	.5153	.6370	4.30
−11	97.31	26.33	08592	.4307	.5324	4.69
−14	94.13	24.90	.07114	.3566	.4408	5.17
−17	91.31	23.58	.05863	.2939	.3633	5.74
−20	88.31	22.33	.04778	.2395	.2960	6.47
−23	85.31	21.15	.03853	.1931	.2387	7.41
−26	82.31	20.04	.03081	.1544	.1909	8.69
−29	79.31	18.99	.02442	.1224	.1513	10.54
−32	76.31	17.99	.01915	.09599	.1187	13.41
−35	73.31	17.03	.01483	.07434	.09189	18.53
−36.027	72.28	16.72	.01364	.06837	.08451	21.28

15. Determination of pressure at separation. Let us put

$$\bar{\mu} = \tfrac{1}{2}(\beta + \mu_T)$$

where μ_T is the virtual mach angle (§13) at the tail vertex of the profile and β is the mach angle of the undisturbed stream. Thus

$$\beta = \sin^{-1}\frac{1}{M} = 28.04 \qquad \text{for } M = 2.13,$$

$$\beta = \sin^{-1}\frac{1}{M} = 32.72 \qquad \text{for } M = 1.85.$$

By assumption (δ) the actual mach angle $\mu = \bar{\mu}$ at the point of separation. *Hence the final value of the pressure ratio p/p_1, which is attained at the separation point by assumption (β), occurs when $\mu = \bar{\mu}$, and hence can be found, in the case of the GU2 and GU3 profiles, by interpolation from the columns for μ and p/p_1 in the tables at the end of* §14.

Since we allow the possibility of pressure variations corresponding both to expansive and compressive flow along the profile (§8), it is important to observe that the relationship between μ and p/p_1 is that given by the tables at the end of §14 for the *GU*2 and *GU*3 profiles irrespective of the type of flow under consideration. But this follows from the fact that along the profile p/p_1 is an explicit function of μ in the general case of non-isentropic flow which we dealt with at the beginning of this paper. To derive this function divide both members of equation (35) by c^2 to obtain

$$\frac{1}{\gamma - 1} + \frac{m^2}{2} = \frac{H}{c^2}\frac{Hc_1^2}{c^2/c_1^2}. \tag{81}$$

But,

$$\frac{H}{c_1^2} = \frac{\gamma p_1}{(\gamma - 1)\rho_1 c_1^2} + \frac{1}{2}\frac{W^2}{c_1^2} = \frac{1}{\gamma - 1} + \frac{M^2}{2}.$$

Making this substitution in the right member of (81) and also the substitution for c^2/c_1^2 given by (54), we find

$$\frac{1}{\gamma - 1} + \frac{m^2}{2} = \frac{[(\gamma - 1)M^2 + 2](\rho/\rho_1)}{2(\gamma - 1)(p/p_1)}, \tag{82}$$

which is valid along the profile. Eliminating p/p_1 from (82) by (53), the resulting equation can be written

$$\left(\frac{p}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{[(\gamma - 1)M^2\ 2]}{(N/N_1)^{1/\gamma}[(\gamma - 1)m^2 + 2]}.$$

Finally taking $\gamma = 1.4$ this equation becomes

$$\frac{p}{p_1} = \frac{(M^2 + 5)^{3.5}}{(N/N_1)^{2.5}(m^2 + 5)^{3.5}} \tag{83}$$

in which the mach number m is to be expressed in terms of μ by the relation $\sin \mu = 1/m$. Hence the relationship between p/p_1 and μ along the profile is the same regardless of whether we are dealing with expansive or compressive flow, and hence the use of the tables at the end of §14, by which this relationship is expressed, is justified in the determination of the final pressure ratio according to the above italicized statement.

For example, in the case of the profile $GU2$ with $M = 2.13$, $a = 0$, we have $\mu_T = 22.84$ from the first of the tabulations at the end of §14. Hence $\bar{\mu} = (28.04 + 22.84)/2 = 25.44$. From the table containing the above value of μ_T we now see that the separation pressure ratio p/p_1, which is the value of p/p_1 for $\mu = 25.44$, lies between .6559 and .7440, and by interpolation we find its value to be .7155.

Let us denote the separation or final pressure on the profile by $\bar{p}$ so that $\bar{p}/p_1$ represents the pressure ratio at separation. Table 6 gives the values of $\bar{\mu}$ and $\bar{p}/p_1$ for the profiles $GU2$ and $GU3$ at the angles of attack a and mach numbers M under consideration.

TABLE 6

Separation pressures

Profile	M	a	$\bar{\mu}$	$\bar{p}/p_1$
$GU2$	2.13	0	25.44	.7155
$GU2$	2.13	4	24.59	.6479
$GU2$	2.13	6	24.18	.6137
$GU2$	2.13	8	23.82	.5822
$GU2$	2.13	10	23.46	.5517
$GU3$	2.13	0	24.21	.5480
$GU3$	2.13	5	23.08	.4916
$GU3$	2.13	10	22.16	.4365
$GU3$	2.13	14	21.54	.3970
$GU3$	1.85	4	26.98	.5456
$GU3$	1.85	16	24.72	.4303

16. Determination of the separation point. Suppose that the flow separates from the profile at a point P, at which the inclination is ω (see Fig. 9). The tangent line PQ will be a stream line by assumption (α) and there will arise a shock at some point Q, the stream line QR behind this shock being parallel to the undisturbed flow by assumption (γ); since we sup-

pose the direction of the undisturbed flow to be along the horizontal, the stream line QR will be horizontal as indicated in Fig. 9. Also by assumption (γ), the shock line QS will make an angle μ with the line QR where μ is the virtual mach angle at P. Now the mach number of the flow will have a constant value $\bar{m}$ along the stream line PQ. This follows from

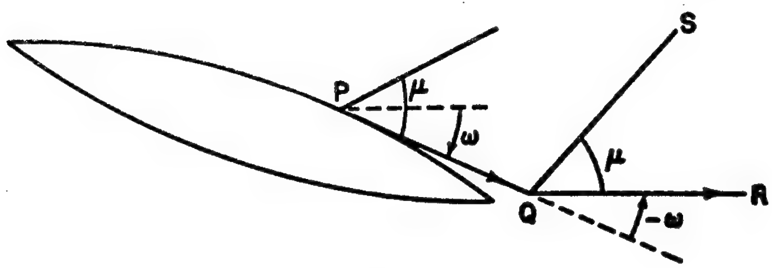


FIG. 9

equation (83), and the fact that the pressure ratio p/p_1 is constant along PQ by assumption (β). Hence, with reference to Fig. 9 and the relation (74), we see immediately that

$$(84)\quad \tan(-\omega) = \frac{2[(\bar{m}^2 - 1)\tan^2(\mu - \omega) - 1]}{[(\gamma - 1)\bar{m}^2 + 2]\tan^3(\mu - \omega) + [(\gamma + 1)\bar{m}^2 + 2]\tan(\mu - \omega)}$$

where we suppose $-\omega \geqq 0$, since separation of flow presumably occurs at a point P at which the inclination ω is not positive under the conditions of the problem.

If we insert the value $\gamma = 1.4$ into (84) and then solve for $\bar{m}^2$, we find

$$(85)\quad \bar{m}^2 = \frac{-5[\tan^3(\mu - \omega)\tan(-\omega) + \tan(\mu - \omega)\tan(-\omega) + \tan^2(\mu - \omega) + 1]}{\tan^3(\mu - \omega)\tan(-\omega) + 6\tan(\mu - \omega)\tan(-\omega) - 5\tan^2(\mu - \omega)}.$$

Equation (85) *is used to determine the entries in the last column of the tables at the end of* §14. Specifically to determine the value of $\bar{m}^2$ corresponding to any value of ω under consideration ($\omega < 0$) in these tables we substitute into the right member of (85) the value of ω in question and the associated value of μ appearing in the table.

Let us think of the quantity $\bar{m}$ as a function of ω in accordance with the above statement concerning its determination. *Then if separation occurs at a point P at which the inclination of the profile is ω, the corre-*

sponding value of $\bar{m}$ must be the value of the mach number at the separation point P. But from assumption (δ), in which $\mu = \bar{\mu}$ (see §15), the value $\bar{m}$ of the mach number at separation is given by $\sin \bar{\mu} = 1/\bar{m}$. This relation enables us to determine $\bar{m}$, and hence we can find the value of the inclination ω at separation from the relation (85). Thus for the profile $GU2$, with $M = 2.13$ and $a = 0$, we have $\bar{\mu} = 25.44$. Hence from $\sin \bar{\mu} = 1/\bar{m}$ we find $\bar{m}^2 = 5.42$. This places the value of ω, at which separation occurs, between -7 and -9, as can be seen from the table for

TABLE 7

Inclination at points of separation

Profile	M	a	$\bar{m}^2$	$\bar{\Omega}$
$GU2$	2.13	0	5.42	−8.78
$GU2$	2.13	4	5.78	−10.87
$GU2$	2.13	6	5.96	−11.88
$GU2$	2.13	8	6.13	−12.83
$GU2$	2.13	10	6.31	−13.78
$GU3$	2.13	0	5.95	−14.14
$GU3$	2.13	5	6.51	−15.63
$GU3$	2.13	10	7.03	−17.31
$GU3$	2.13	14	7.42	−18.49
$GU3$	1.85	4	4.86	−13.09
$GU3$	1.85	16	5.72	−16.89

$GU2$, $M = 2.13$, $a = 0$ in §14. Hence by interpolation, using the columns for ω and $\bar{m}^2$ in this table, we find that separation occurs at $\omega = -8.78$. Table 7 gives the values of the inclination $\bar{\Omega}$ at which separation takes place for the profiles, mach numbers M, and angles of attack a under consideration.

17. Origin of back pressure. We have now found the pressure ratio $\bar{p}/p_1$ (§15) and also the inclination $\bar{\Omega}$ of the profile at the separation point (§16). These values $\bar{\Omega}$ and $\bar{p}/p_1$ determine the pressure ratio p/p_1 at points preceding the separation point by the method of §9. The graph of the ratio p/p_1 so determined for the $GU2$ and $GU3$ profiles must meet the previous graph of this ratio, as determined in §14, so that the two graphs will combine to define p/p_1 as a continuous function along the profile, on account of the continuity requirement of §12. But for these graphs to intersect, we must apply the formula (45) for compressive flow in the above calculation of the ratio p/p_1. In other words, for the calculation of the ratio p/p_1 immediately preceding the separation point,

we must use the formulas

$$f(\mu) = f(\bar{\mu}) + \bar{\Omega} - \omega, \tag{86}$$

$$\frac{p}{\bar{p}} = \frac{g(\mu)}{g(\bar{\mu})}, \qquad \frac{p}{p_1} = \frac{p}{\bar{p}} \cdot \frac{\bar{p}}{p_1}, \tag{87}$$

where the values of $\bar{\mu}$, $\bar{\Omega}$, and $\bar{p}/p_1$ are known from §§15 and 16. Calculation of the ratio p/p_1 by the formulas (86) and (87) for the $GU2$ and $GU3$ profiles under consideration now leads to the following tables:

$GU2$, $M = 2.13$, $a = 0$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−8.78	95.46	25.44	.07662	1	.7155
−8	94.68	25.07	.07283	.9505	.6801
−7	93.68	24.62	06843	.8931	.6390
−6	92.68	24.17	.06408	.8363	.5984

$GU2$, $M = 2.13$, $a = 4$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−10.87	93.61	24.59	.06814	1	.6479
−10	92.74	24.20	.06437	.9447	.6121
−9	91.74	23.76	.06026	.8844	.5730
−8	90.74	23.33	.05640	.8277	.5363
−7	89.74	22.91	.05263	.7724	.5004
−6	88.74	22.50	.04920	.7220	.4678
−5	87.74	22.10	.04586	.6730	.4360

$GU2$, $M = 2.13$, $a = 6$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−11.88	92.70	24.18	.06417	1	.6137
−11	91.82	23.79	.06053	.9433	.5789
−10	90.82	23.37	.05673	.8841	.5426
−9	89.82	22.94	.05288	.8241	.5058
−8	88.82	22.54	.04953	.7719	.4737
−7	87.82	22.13	.04611	.7186	.4410
−6	86.82	21.74	.04304	.6707	.4116

$GU2$, $M = 2.13$, $a = 8$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−12.83	91.88	23.82	.06080	1	.5822
−12	91.05	23.47	.05763	.9479	.5519
−11	90.05	23.04	.05374	.8839	.5146
−10	89.05	22.63	.05029	.8271	.4815
−9	88.05	22.22	.04686	.7707	.4487
−8	87.05	21.83	.04372	.7191	.4187
−7	86.05	21.44	.04075	.6702	.3902

$GU2$, $M = 2.13$, $a = 10$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−13.78	91.04	23.46	.05754	1	.5517
−13	90.26	23.13	.05456	.9482	.5231
−12	89.26	22.72	.05104	.8870	.4894
−11	88.26	22.31	.04761	.8274	.4565
−10	87.26	21.91	.04433	.7704	.4250
−9	86.26	21.52	.04136	.7188	.3966
−8	85.26	21.14	.03846	.6684	.3688

$GU3$, $M = 2.13$, $a = 0$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−14.14	92.77	24.21	.06446	1	.5480
−14	92.63	24.15	.06388	.9910	.5431
−13	91.63	23.71	.05981	.9279	.5085
−12	90.63	23.29	.05600	.8688	.4761
−11	89.63	22.87	.05229	.8112	.4445
−10	88.63	24.46	.04887	.7581	.4154

$GU3$, $M = 2.13$, $a = 5$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−15.63	90.15	23.08	.05410	1	.4916
−15	89.52	22.82	.05188	.9590	.4714
−14	88.52	22.41	.04845	.8956	.4403
−13	87.52	22.01	.04510	.8336	.4098
−12	86.52	21.62	.04212	.7786	.3828
−11	85.52	21.24	.03922	.7250	.3564

*GU*3, $M = 2.13$, $a = 10$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−17.31	87.89	22.16	.04636	1	.4365
−17	87.58	22.03	.04527	.9765	.4262
−16	86.58	21.64	.04227	.9118	.3980
−15	85.58	21.26	.03937	.8492	.3707
−14	84.58	20.88	.03657	.7888	.3443
−13	83.58	20.51	.03403	.7340	.3204

*GU*3, $M = 2.13$, $a = 14$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−18.49	86.31	21.54	.04151	1	.3970
−18	85.82	21.35	.04006	.9651	.3831
−17	84.82	20.97	.03718	.8957	.3556
−16	83.82	20.60	.03465	.8348	.3314
−15	82.82	20.23	.03212	.7738	.3072
−14	81.82	19.87	.02975	.7167	.2845

*GU*3, $M = 1.85$, $a = 4$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−13.09	98.61	26.98	.09291	1	.5456
−13	98.52	26.94	.09248	.9954	.5431
−12	97.52	26.44	.08710	.9375	.5115
−11	96.52	25.94	.08175	.8799	.4801
−10	95.52	25.47	.07693	.8280	.4518
−9	94.52	25.00	.07211	.7761	.4234

*GU*3, $M = 1.85$, $a = 16$

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−16.89	93.91	24.72	.06940	1	.4303
−16	93.02	24.32	.06553	.9442	.4063
−15	92.02	23.88	.06134	.8839	.3803
−14	91.02	23.45	.05745	.8278	.3562
−13	90.02	23.03	.05365	.7731	.3327

To illustrate the procedure for determining the point on the profile at which the two graphs for p/p_1 intersect, let us consider the *GU*2 profile

with $M = 2.13$ and $a = 0$. Let $\omega = \Omega$ be the value of ω at which the graphs intersect, i.e., at which the ratio p/p_1 has the same value for each graph. Then, from the above table for $GU2$, $M = 2.13$, $a = 0$, and the corresponding table in §14, we see that Ω must lie between -7 and -8, the value of p/p_1 being .6155 at $\omega = -8$ from the table in §14. Hence if we put $\Omega = -(7 + \delta)$ so that δ is a decimal fraction, we must have

$$.6559 - .0404\,\delta = .6390 + .0411\,\delta,$$

since the left member of this equation is the value of p/p_1 at $\omega = \Omega$, as found by interpolation from the table in §14, while the right member is the interpolated value of p/p_1 at $\omega = \Omega$ from the table in this section. Solving for δ we find $\delta = .21$, and hence $\Omega = -7.21$. The calculated values of Ω for the various cases under consideration are given in Table 8.

It is thus seen that the pressure ratio p/p_1 will be determined along the profile according to the calculations of §14 for expansive flow until the point $\omega = \Omega$ is reached. Between $\omega = \Omega$ and the separation value $\omega = \bar{\Omega}$ is an interval of increasing pressure or *back pressure interval* following which the flow leaves the profile along the tangent and the pressure remains constant and equal to its value $\bar{p}$ at the separation point. It is interesting to observe from the above discussion that *the back pressure has its origin essentially in the requirement of continuity stated in* §12.

TABLE 8

Profile	M	a	Ω
*GU*2	2.13	0	−7.21
*GU*2	2.13	4	−8.98
*GU*2	2.13	6	−9.87
*GU*2	2.13	8	−10.77
*GU*2	2.13	10	−11.64
*GU*3	2.13	0	−12.29
*GU*3	2.13	5	−13.58
*GU*3	2.13	10	−15.17
*GU*3	2.13	14	−16.39
*GU*3	1.85	4	−12.02
*GU*3	1.85	16	−15.63

18. Complete pressure calculations. We have now completed all calculations for the determination of pressure on the $GU2$ and $GU3$ profiles under consideration. It remains to arrange our results in such form that the calculated pressure graphs can readily be drawn and compared with

the corresponding experimental graphs. For this purpose we shall now replace the inclination ω, which has served as our independent variable defining position on the profile, by the % chord δ. It can easily be verified that the relation between δ, ω and the angle of attack a is given by

$$\sin(\omega + a) = .2 - \frac{\delta}{250} \qquad \text{for the } GU2 \text{ profile,}$$

$$\sin(\omega + \alpha) = \frac{50 - \delta}{146} \qquad \text{for the } GU3 \text{ profile.}$$

By means of these relations the % chord δ, corresponding to any given inclination ω and angle of attack a, is determined; conversely when δ and a are assigned, the inclination ω can be found, and hence the value of the pressure ratio p/p_1 by interpolation, using the appropriate table in §14 or §17. We can thus construct the following tables for the pressure ratios p/p_1 in which the values of $\bar{\Omega}$ and Ω are given by Table 7 and Table 8, respectively.

$GU2$, $M = 2.13$, $a = 0$

% chord	ω	p/p_1
0	11.54	1.891
10	9.21	1.675
20	6.89	1.481
30	4.59	1.283
40	2.29	1.149
50	0	1.008
60	−2.29	.880
70	−4.59	.764
80	−6.89	.661
81.38	Ω	.647
84	−7.82	.673
86	−8.28	.693
88.16	$\bar{\Omega}$	.716
90		.716
100		.716

GU2, $M = 2.13$, $a = 4$

% chord	ω	p/p_1
0	7.54	1.529
10	5.21	1.347
20	2.89	1.183
30	.59	1.036
40	−1.71	.905
50	−4.00	.786
60	−6.29	.681
70	−8.59	.587
71.70	Ω	.572
74	−9.51	.593
77	−10.20	.620
79.90	$\bar{\Omega}$	.648
80		.648
90		.648
100		.648

GU2, $M = 2.13$, $a = 6$

% chord	ω	p/p_1
0	5.54	1.371
10	3.21	1.204
20	.89	1.054
30	−1.41	.920
40	−3.71	.800
50	−6.00	.692
60	−8.29	.598
66.87	Ω	.538
70	−10.59	.564
73	−11.28	.590
75.61	$\bar{\Omega}$	.614
80		.614
90		.614
100		.614

*GU*2, $M = 2.13$, $a = 8$

% chord	ω	p/p_1
0	3.54	1.226
10	1.21	1.075
20	−1.11	.938
30	−3.41	.816
40	−5.71	.706
50	−8.00	.610
60	−10.29	.524
62.08	Ω	.507
65	−11.44	.531
68	−12.13	.557
71.50	$\bar{\Omega}$	.582
80		.582
90		.582
100		.582

*GU*2, $M = 2.13$, $a = 10$

% chord	ω	p/p_1
0	1.54	1.093
10	−.79	.954
20	−3.11	.830
30	−5.41	.719
40	−7.71	.620
50	−10.00	.533
57.15	Ω	.477
60	−12.29	.499
63	−12.98	.522
66.48	$\bar{\Omega}$	.552
70		.552
80		.552
90		.552
100		.552

$GU3$, $M = 2.13$, $a = 0$

% chord	ω	p/p_1
0	20.03	2.915
10	15.90	2.388
20	11.86	1.954
30	7.87	1.591
40	3.93	1.292
50	0	1.039
60	−3.93	.824
70	−7.87	.649
80	−11.86	.502
81.08	Ω	.486
83	−13.06	.511
84	−13.47	.525
85.67	$\bar{\Omega}$	.548
90		.548
100		.548

$GU3$, $M = 2.13$, $a = 5$

% chord	ω	p/p_1
0	15.03	2.262
10	10.90	1.839
20	6.86	1.488
30	2.87	1.194
40	−1.07	.954
50	−5.00	.754
60	−8.93	.585
70	−12.87	.451
71.78	Ω	.428
73	−14.06	.442
75	−14.86	.467
76.93	$\bar{\Omega}$	.492
80		.492
90		.492
100		.492

GU3, $M = 2.13$, $a = 10$

% chord	ω	p/p_1
0	10.03	1.747
10	5.90	1.405
20	1.86	1.121
30	−2.13	.885
40	−6.07	.696
50	−10.00	.538
60	−13.93	.410
63.16	Ω	.376
65	−15.90	.395
67	−16.69	.418
68.58	$\bar{\Omega}$	.436
70		.436
80		.436
90		.436
100		.436

GU3, $M = 2.13$, $a = 14$

% chord	ω	p/p_1
0	6.03	1.409
10	1.90	1.121
20	−2.14	.884
30	−6.13	.688
40	−10.07	.534
50	−14.00	.407
56.09	Ω	.342
58	−17.14	.359
60	−17.93	.381
61.43	$\bar{\Omega}$	.397
70		.397
80		.397
90		.397
100		.397

GU3, $M = 1.85$, $a = 4$

% chord	ω	p/p_1
0	16.03	2.264
10	11.90	1.850
20	7.86	1.517
30	3.87	1.229
40	−.07	1.012
50	−4.00	.818
60	−7.93	.653
70	−11.87	.517
70.37	Ω	.512
71	−12.27	.520
72	−12.67	.533
73.07	$\bar{\Omega}$	.546
80		.546
90		.546
100		.546

GU3, $M = 1.84$, $a = 16$

% chord	ω	p/p_1
0	4.03	1.236
10	−.10	.997
20	−4.14	.798
30	−8.13	.633
40	−12.07	.500
49.06	Ω	.397
50	−16.00	.406
51	−16.39	.417
52.57	$\bar{\Omega}$	.430
60		.430
70		.430
80		.430
90		.430
100		.430

19. Comparison of theoretical and experimental pressure graphs for the GU2 and GU3 profiles. The experimental values of p/p_1 obtained by Ferri [1] for the $GU2$ and $GU3$ profiles under consideration are given in Tables 9, 10 and 11. In the actual experiments the data contained in Table 10 were obtained by measurements on the under side of the $GU2$

TABLE 9

Exp. p/p_1 *for GU2,* $M = 2.13$

% chord	$a = 0$	$a = 4$	$a = 8$
4	1.825	1.468	1.153
12.5	1.557	1.258	.981
21.5	1.384	1.105	.875
31.0	1.240	.962	.732
40.5	1.096	.895	.655
50.5	.971	.790	.598
59.0	.856	.675	.540
68.5	.741	.618	.559
78.7	.645	.683	.598
87.5	.702	.694	.598
97.0	.702	.694	.598

TABLE 10

Exp. p/p_1 *for GU2,* $M = 2.13$

% chord	$a = 6$	$a = 10$
13.6	1.134	.904
22.5	.981	.808
31.5	.866	.713
40.5	.790	.646
50.5	.675	.550
59.0	.598	.521
68.5	.579	.559
78.7	.637	.559
87.5	.637	.559

profile at angles of attack $a = -6$ and $a = -10$; we have, however, listed this data as though obtained from measurements on the upper contour of the profile for angles $a = 6$ and $a = 10$ to conform to our previous notation.

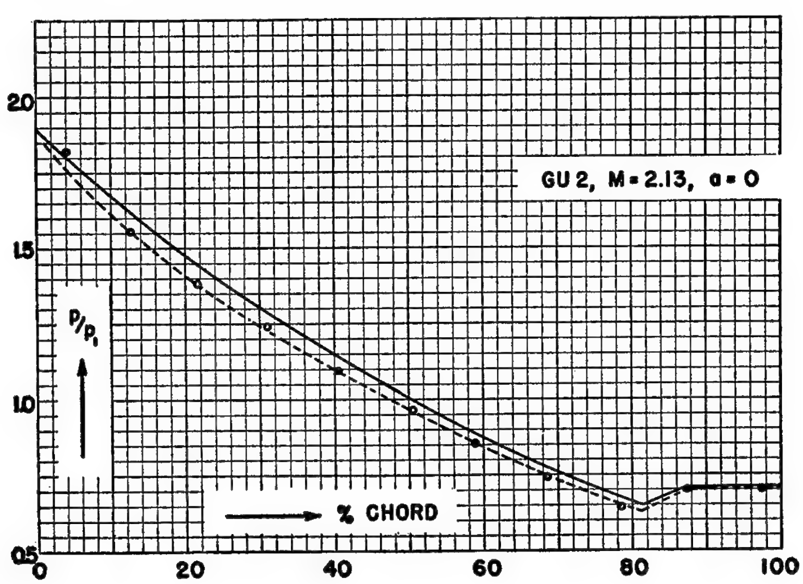
In the graphs of this section the dotted lines are the experimental pressure curves based on the data in Tables 9, 10 and 11; the small circles on these graphs indicate the actual experimental values of the

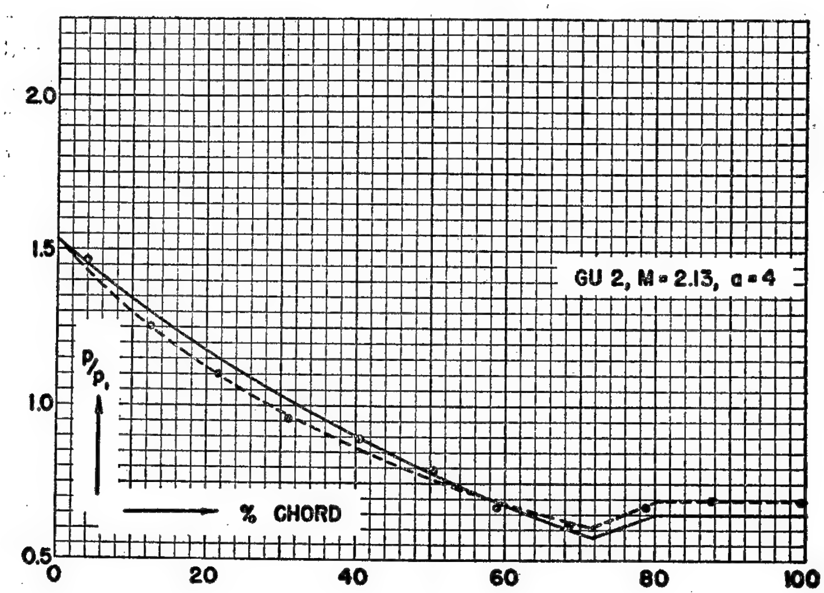
ratio p/p_1. The heavy curves are the pressure graphs determined from the calculated values of the pressure ratio p/p_1 in the tables at the end of §18. It is seen that the agreement between the experimental and calculated pressure graphs is excellent. In general, the deviation between the calculated and experimental values of p/p_1 at the rear is very closely equal to the difference between these values at a point at the beginning

TABLE 11

Exp. p/p_1 *for GU3*

% chord	$M = 2.13$				$M = 1.85$	
	$\alpha = 0$	$\alpha = 5$	$\alpha = 10$	$\alpha = 14$	$\alpha = 4$	$\alpha = 16$
5	2.481	1.871	1.531	1.213	1.944	1.137
11.5	2.146	1.594	1.383	1.000	1.714	1.000
20.5	1.784	1.361	1.017	.789	1.466	.826
30	1.488	1.127	.808	.660	1.199	.677
40	1.244	.945	.669	.530	1.006	.590
49	1.042	.746	.564	.443	.863	.516
59.2	.873	.608	.460	.408	.714	.590
68.4	.763	.486	.443	.443	.590	.590
77.5	.634	.561	.486	.443	.609	.590
86.2	.681	.561	.486	.443	.609	.590
95	.681	.561	.490	.443	.609	.590

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p/p₁
% CHORD
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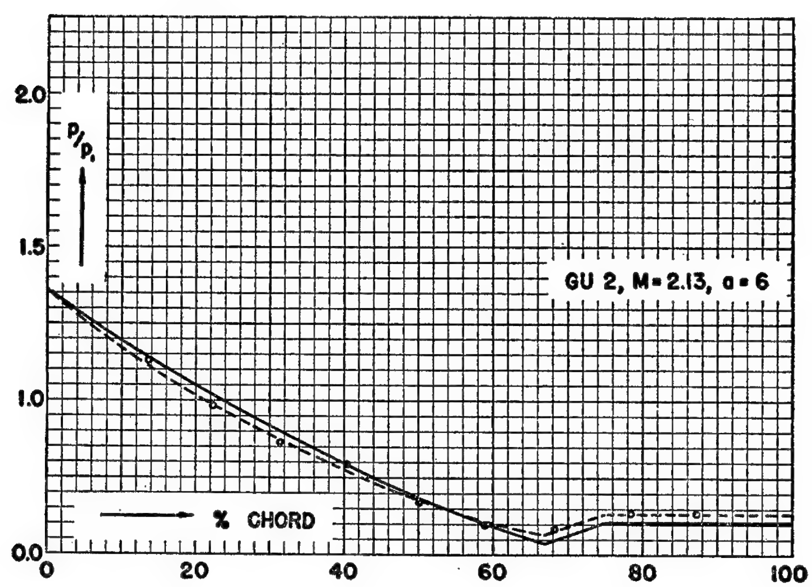
1.5
1.0
0.5
0
20
40
60
80
100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p/p₁
% CHORD
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0
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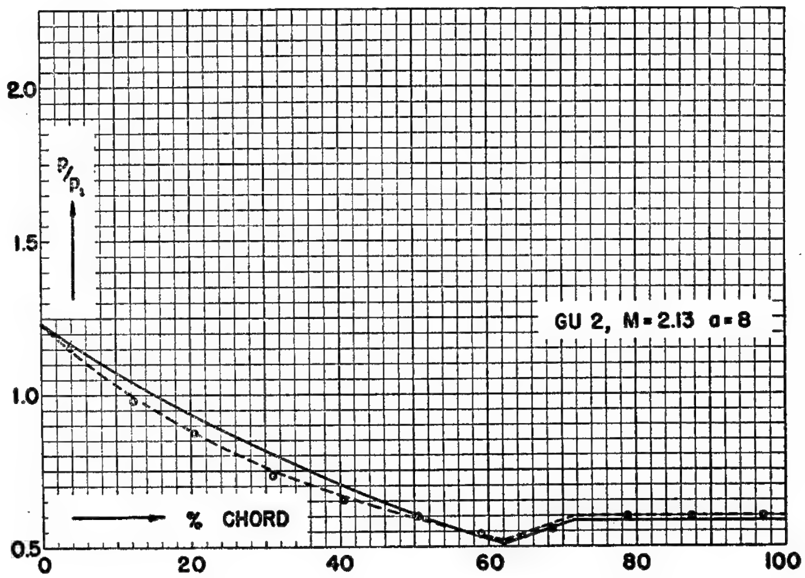
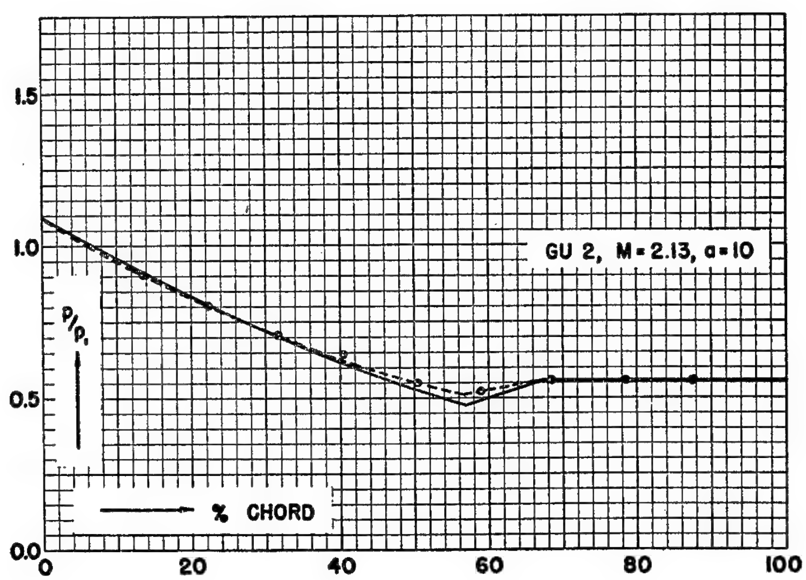
40
60
80
100


2.0
1.5
1.0
0.5
p/p1
GU 2, M = 2.13 α = 8
% CHORD
0
20
40
60
80
100


1.5
1.0
0.5
0.0
p/p1
GU 2, M = 2.13, α = 10
% CHORD
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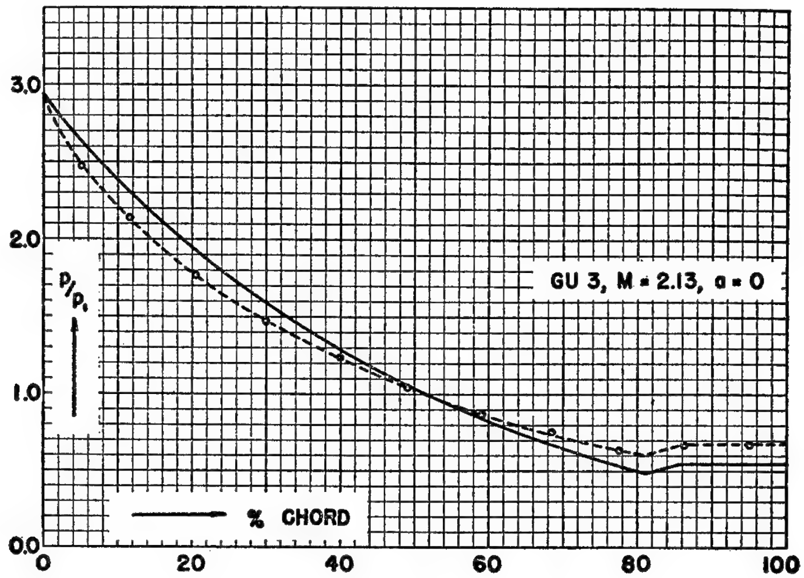
20
40
60
80
100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p/p₁
% CHORD
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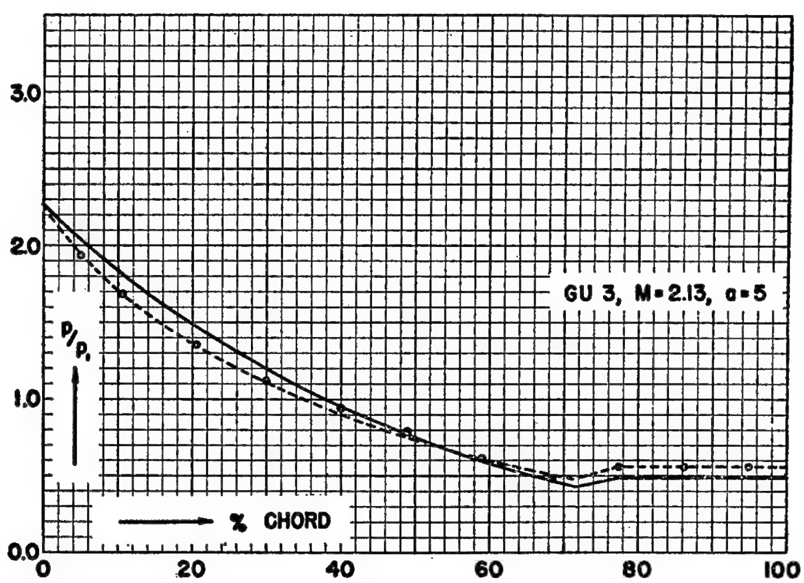
1.0
0.0
0
20
40
60
80
100


GU 3, M = 2.13, α = 5
p/p₁
% CHORD
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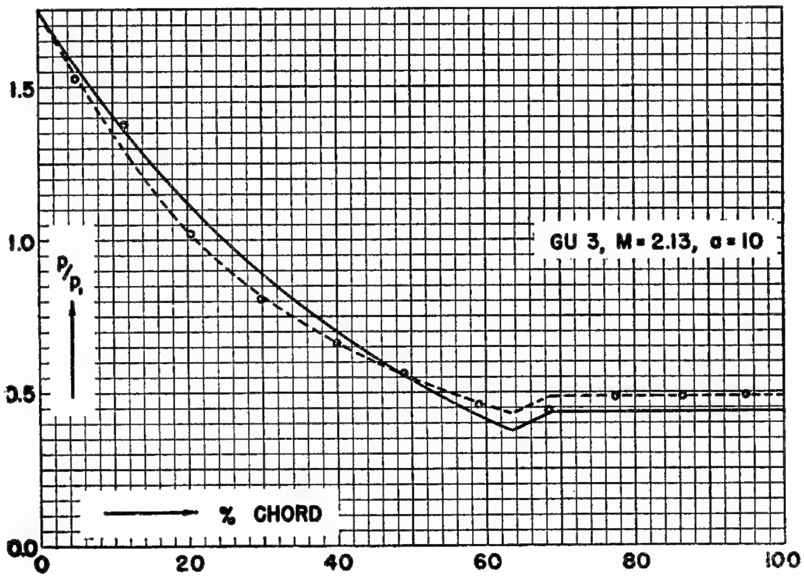
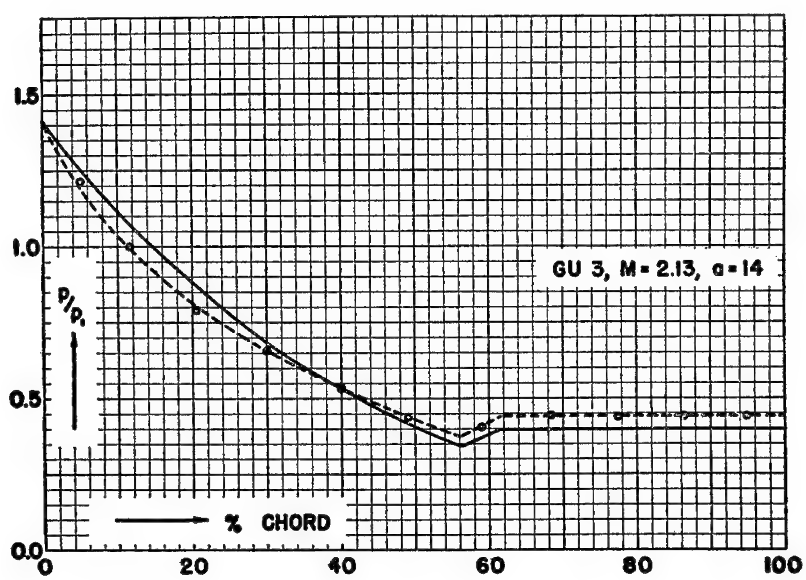
2.0
1.0
0.0
0
20
40
60
80
100


1.5
1.0
0.5
0.0
p/p_1
GU 3, M = 2.13, α = 10
% CHORD
0
20
40
60
80
100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0.0
p/p_1
GU 3, M = 2.13, α = 14
% CHORD
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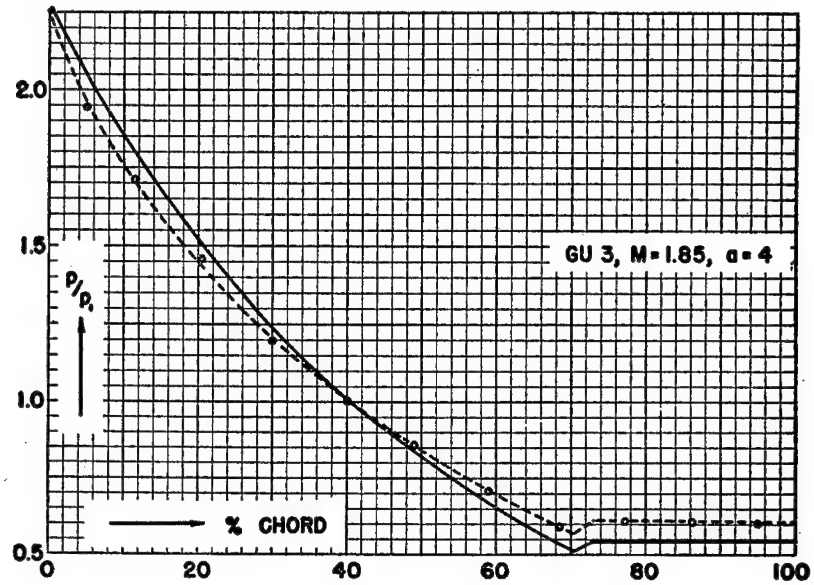
20
40
60
80
100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p/p1
% CHORD
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1.0
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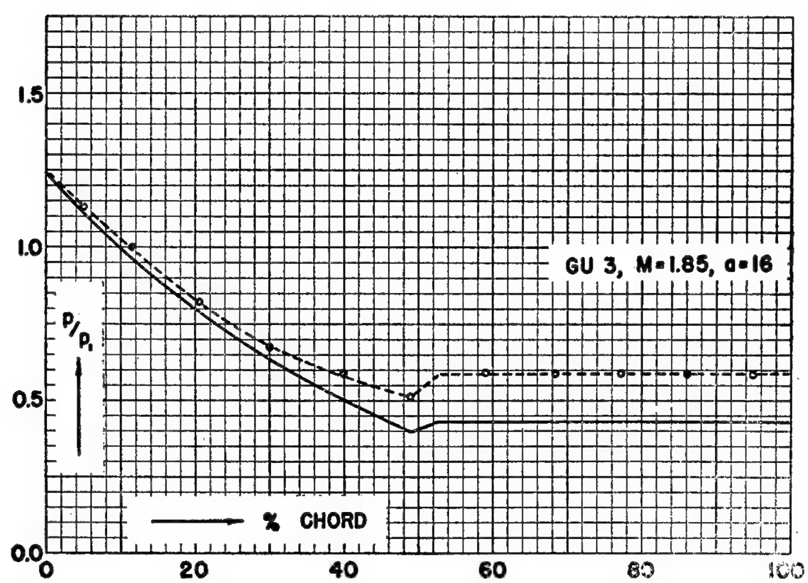
0
20
40
60
80
100


GU 3, M=1.85, a=16
p/p1
% CHORD
1.5
1.0
0.5
0.0
0
20
40
60
80
100

of the interval in which back pressure becomes effective. This suggests, as mentioned in the introduction, that an improvement in the standard pressure calculation of §14 will result in a corresponding improvement in the agreement, at the rear of the profile, between experimental and calculated values of the pressure. We shall examine this hypothesis more fully in the following section.

20. A direct test of the separation theory with experiment. To test the hypothesis mentioned at the end of §19, let us extend the experimental pressure graph for $GU3$, $M = 2.13$, $a = 0$ in a natural way so as to obtain the curve shown at the top of the graph appearing in this section. This extended curve will now replace, in the separation calculation, the theoretical pressure curve given by the standard procedure (§14) and may in fact be considered as an improved version of the latter curve on account of its close relation to the experimental data. We must now determine the various quantities appearing in Table 12. The first four of the % chords and the associated pressure ratios p/p_1 in this table are the same as have already appeared in Table 11 of experimental values of p/p_1 for $GU3$, $M = 2.13$, $a = 0$ in §19; however the last three % chords and the corresponding pressure ratios are found from the extended part of the experimental pressure curve. Values of ω in the second column of Table 12 are determined by the formula

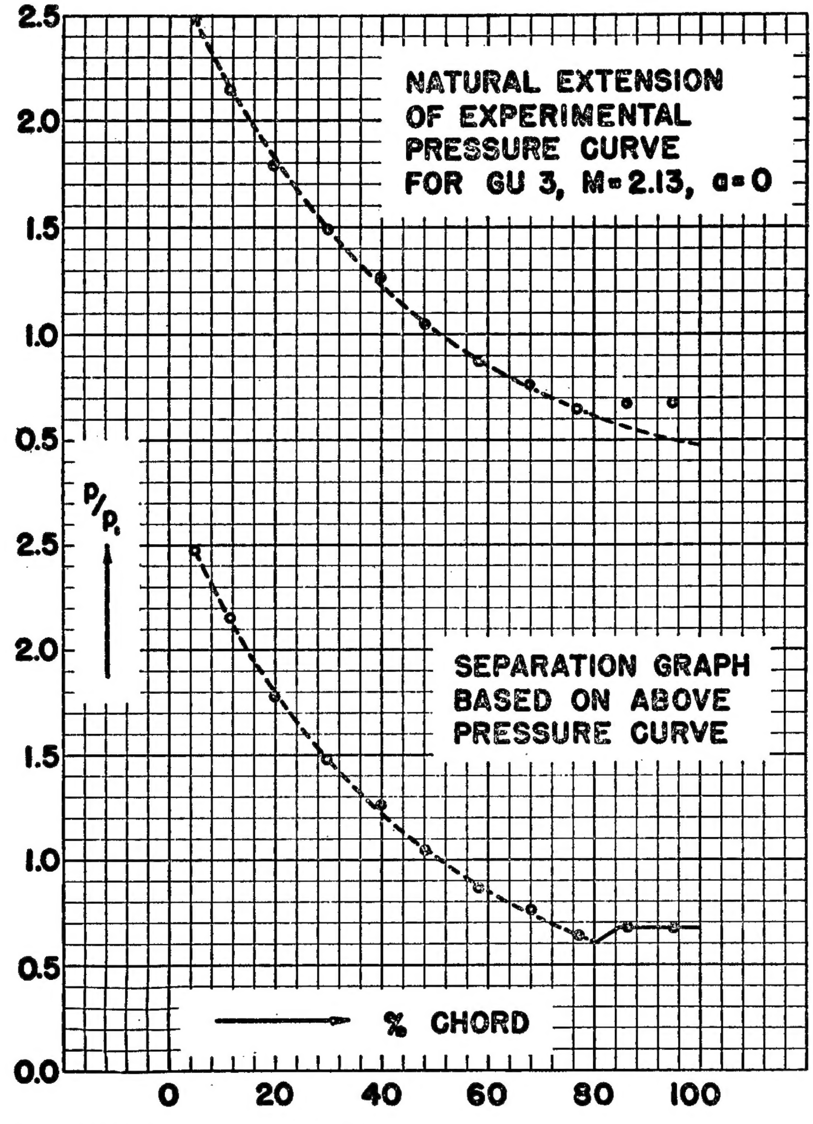
$$\delta = 50 - 146 \sin \omega$$

which is obtained from the corresponding formula in §18 by taking $a = 0$. To calculate the column of values of the ratio ρ/ρ_1 in Table 12, we have the formula (53), namely

$$\frac{\rho}{\rho_1} = \left[\frac{(p/p_1)}{(N/N_1)}\right]^{1/1.4},$$

TABLE 12

% chord	ω	p/p_1	ρ/ρ_1	$(v/W)^2$	m	μ	m^2
49	.39	1.042	.9947	.9476	2.0258	29.58	
59.2	−3.61	.873	.8766	1.0045	2.1392	27.87	4.38
68.4	−7.24	.763	.7962	1.0460	2.2253	26.70	4.60
77.5	−10.86	.634	.6975	1.1003	2.3435	25.01	5.07
86	−14.27	.550	6302	1.1403	2.4347	24.25	5.38
94	−17.54	.500	.5887	1.1661	2.4958	23.62	5.77
100	−20.027	.475	.5675	1.1796	2.5286	23.30	6.11



where N/N_1 is given by (52). Thus

$$\frac{N}{N_1} = \frac{2.9152}{(2.0741)^{1.4}} = 1.0498,$$

using the initial determinations of p/p_1 and ρ/ρ_1 given in Table 4. After the ratios ρ/ρ_1 have been found, the values of $(v/W)^2$ in Table 12 are determined by equation (79), then m is obtained from (80), μ is deter-

mined by the relation $\sin \mu = 1/m$, and finally $\bar{m}^2$ is found by means of equation (85). This completes the determination of all entries in Table 12.

The remainder of the separation calculation is the same as in the preceding discussion. Thus μ_T has the value 23.30 from Table 12, and hence

$$\bar{\mu} = \tfrac{1}{2}(28.04 + 23.30) = 25.67 .$$

Using Table 12 we now find

$$\frac{\bar{p}}{p_1} = .684, \qquad \bar{m}^2 = 5.33, \qquad \bar{\Omega} = -13.72.$$

To determine the value of Ω we use formulas (86) and (87) to set up Table 13. Comparison of the values of p/p_1 in this table with those in Table 12 then leads, as explained in §17, to the value $\Omega = -11.87$.

TABLE 13

ω	$f(\mu)$	μ	$g(\mu)$	$p/\bar{p}$	p/p_1
−13.72	95.94	25.67	.07898	1	.684
−13	95.22	25.33	.07550	.9559	.654
−12	94.22	24.86	.07075	.8958	.613
−11	93.22	24.41	.06664	.8438	.577

Table 14 giving the final form of the calculated values of p/p_1 can now be constructed. Entries in this table for % chords less than 80.03, corresponding to $\omega = \Omega$, have been omitted, since this part of the pressure graph is obtained from the experimental values of p/p_1. Using Table 14 we can now plot the pressure curve for $GU3$, $M = 2.13$, $a = 0$, which is shown at the bottom of the graph in this section. It is observed that agreement between experimental and calculated back pressure intervals and final or separation pressures is remarkably accurate.

TABLE 14

% chord	ω	p/p_1
80.03	Ω	.608
81.5	−12.46	.632
83	−13.06	.656
84.63	$\bar{\Omega}$	.684
90		.684
100		.684

References

1. A. Ferri, *Alcuni Risultati sperimentali riguardanti profili alari provati alla galleria ultrasonora di Guidonia*, Atti di Guidonia, No. 17, 1939, pp. 337–372. Translated as N.A.C.A. Technical Memorandum No. 946.
2. See T. Y. Thomas, *The determination of pressure on curved bodies behind shocks*, Communications on Pure and Applied Mathematics, New York Univ. Inst. for Math. and Mech., Vol. 3, 1950, pp. 103–132.
3. Cp. Edmonson, Murnaghan and Snow, *The theory and practice of two-dimensional supersonic pressure calculations*, The Johns Hopkins Univ. Ap. Phys. Lab., Bumblebee Report No. 26, 1945.
4. For a derivation of this form of the shock conditions, see *The fundamental hydrodynamical equations and shock conditions for gases*, Math. Mag., vol. 22, 1949, pp. 169–189, and *Calculation of the curvatures of attached shock waves*, Jour. of Math. and Phys., vol. 27, 1949, pp. 279–297.